R. Bruner
Math 2250, Fall 2008, Quiz 8
Halloween, 2008

Let P_2 be the vector space of polynomials whose degree is 2 or less.

1. Show that $\{(x-1)^2, x^2, (x+1)^2\}$ is linearly independent.
2. Show that this set of polynomials spans P_2.

1. If $r_1(x-1)^2 + r_2 x^2 + r_3(x+1)^2 = 0$ then

5

at $x=0$ we get $r_1 + r_3 = 0$
at $x=1$ we get $0 + r_2 + 4r_3 = 0$
at $x=-1$ we get $4r_1 + r_2 = 0$.

Any 3 points would work for this.

Together we have $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 4 \\ 4 & 1 & 0 \end{bmatrix}\begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

Row reducing $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 4 \\ 4 & 1 & 0 \end{bmatrix} \xrightarrow[-4R_1]{} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 4 \\ 0 & 1 & -4 \end{bmatrix} \xrightarrow[-R_2]{} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 4 \\ 0 & 0 & -8 \end{bmatrix}$

we have 3 pivots, so $r_1 = r_2 = r_3 = 0$. //

2. Proof 1: $\dim P_2 = 3$ and $\{(x-1)^2, x^2, (x+1)^2\}$ is a linearly independent set in P_2, hence it spans.

5

Proof 2: To solve $r_1(x-1)^2 + r_2 x^2 + r_3(x+1)^2 = a_0 + a_1 x + a_2 x^2$, for a polynomial $a_0 + a_1 x + a_2 x^2$, we expand the squares to get $(r_1 + r_3) + (-2r_1 + 2r_3)x + (r_1 + r_2 + r_3)x^2 = a_0 + a_1 x + a_2 x^2$.

This gives $\begin{bmatrix} 1 & 0 & 1 \\ -2 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}$. Row reducing,

$\begin{bmatrix} 1 & 0 & 1 \\ -2 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix} \xrightarrow[-R_1]{+2R_1} \begin{bmatrix} \underline{1} & 0 & 1 \\ 0 & 0 & \underline{4} \\ 0 & \underline{1} & 0 \end{bmatrix}$ This has three pivots, so we can solve for r_1, r_2, r_3 for any a_0, a_1, a_2. //