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Math 2250, Fall 2008, Quiz 7
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Let

$$P = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

1. Compute the rank of P.
2. Find a basis for $\mathrm{Im}(P)$.
3. Find a basis for $\mathrm{Ker}(P)$.
4. (Extra credit) Show that, as a transformation from $\mathrm{Im}(P)$ to $\mathrm{Im}(P)$, P is invertible. (Hint: What does it do to the basis you found for $\mathrm{Im}(P)$?)

$$\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \xrightarrow{+R_3} \begin{bmatrix} 1 & 1 & 2 \\ 2 & -1 & 1 \\ 0 & 3 & 3 \end{bmatrix} \xrightarrow[* \frac{1}{3}]{-2R_1} \begin{bmatrix} 1 & 1 & 2 \\ 0 & -3 & -3 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow[+3R_3]{-R_3} \begin{array}{c} \begin{array}{ccc} * & * & \end{array} \\ \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

1. rank $P = 2$

2. $\mathrm{Im}(P)$ has basis $\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$

3. $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ goes to $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ if $\left\{ \begin{matrix} x+z=0 \\ y+z=0 \end{matrix} \right\}$ so $\begin{matrix} x = -z \\ y = -z \\ z = z \end{matrix}$

$\mathrm{Ker}(P) = \mathrm{Span} \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$ (and this is a basis).

4. $P\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ -3 \\ 3 \end{bmatrix} = 3\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$

$P\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ 6 \\ 3 \end{bmatrix} = 3\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$

So P stretches each vector in $\mathrm{Im}(P)$ by a factor of 3: $Pv = 3v$.

The inverse is therefore $v \longmapsto \frac{1}{3}v$.