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Let $W = \text{Span}\left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\}$

1. Find a basis for $W^{\perp}$.

2. Decompose $b = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ into orthogonal components b_W and $b_{W^\perp}$.

1. $\begin{bmatrix} 2 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 1 \end{bmatrix} \xrightarrow[-2R_1]{} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 3 \end{bmatrix}.$ Ker solves $\begin{matrix} x - z = 0 \\ y + 3z = 0 \end{matrix}$

or $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = z \begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix}$

so $W^{\perp} = \text{Span} \begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix}.$

2. $\left[\begin{array}{ccc|c} 2 & 1 & 1 & 1 \\ 1 & 0 & -3 & 1 \\ 1 & -1 & 1 & 1 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -3 & 1 \\ 2 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \end{array}\right] \xrightarrow[-R_1]{-2R_1} \left[\begin{array}{ccc|c} 1 & 0 & -3 & 1 \\ 0 & 1 & 7 & -1 \\ 0 & -1 & 4 & 0 \end{array}\right] \xrightarrow[+R_2]{}$

$\left[\begin{array}{ccc|c} 1 & 0 & -3 & 1 \\ 0 & 1 & 7 & -1 \\ 0 & 0 & 11 & -1 \end{array}\right] \xrightarrow[* \frac{1}{11}]{} \left[\begin{array}{ccc|c} 1 & 0 & -3 & 1 \\ 0 & 1 & 7 & -1 \\ 0 & 0 & 1 & -1/11 \end{array}\right] \xrightarrow[-7R_3]{+3R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 8/11 \\ 0 & 1 & 0 & -4/11 \\ 0 & 0 & 1 & -1/11 \end{array}\right]$

Hence $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}_{W^{\perp}} = -\frac{1}{11} \begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix} = \frac{1}{11} \begin{bmatrix} -1 \\ 3 \\ -1 \end{bmatrix}$

and $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}_{W} = \frac{8}{11} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} - \frac{4}{11} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \frac{1}{11} \begin{bmatrix} 12 \\ 8 \\ 12 \end{bmatrix}$

(also $= \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}_{W^{\perp}} = \frac{1}{11} \begin{bmatrix} 11 - (-1) \\ 11 - 3 \\ 11 - (-1) \end{bmatrix} =$)

COMMENTS

1. We know $\dim W^{\perp} = \dim \mathbb{R}^3 - \dim W = 3-2=1$

so we will have 1 element in the basis of $W^{\perp}$.

2.

Row reducing $\left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & -3 & 0 & 1 & 0 \\ 1 & -1 & 1 & 0 & 0 & 1 \end{array}\right]$ would give us

decompositions $e_i = (e_i)_W + (e_i)_{W^{\perp}}$ for all three

standard basis vectors e_1, e_2 and e_3. okay, but

unnecessary. Then

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}_W = (e_1)_W + (e_2)_W + (e_3)_W, \text{ etc.}$$

3. We can test our basis element for $W^{\perp}$ by taking its dot product with $\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$:

$$\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix} = 2-3+1 = 0 \quad \text{and} \quad \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix} = 1+0-1 = 0$$

So, it would be clear that $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is not in $W^{\perp}$ since

$\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 2+1+1 = 4 \neq 0$ $\left(\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 0, \text{ so it is } \perp \text{ to } \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right),$

so it is not $\perp$ to all of W.