1.

5 $A = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 0 & 1 \end{bmatrix}$

2. (a) Linear : ev_0 and ev_1

(b) Linear : add poly's coordinatewise; scalar mult too.

5 ea (c) Not linear : $2x - 1 \neq 2(x-1)$

3. (a) Not a subspace : $cx + cy = c$, not 1

5 ea (b) Subspace $\left(= \ker \begin{bmatrix} 1 & 1 & -1 \end{bmatrix}\right)$

(c) Subspace : $\ker(ev_1 - 2\, ev_0)$

4.

5 $$\begin{bmatrix} 3 & 2 & 1 \\ 5 & -1 & -1 \\ 4 & 4 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$$

5.

5 $$\begin{bmatrix} 1 & 3 & 0 & 3 \\ 0 & 7 & 2 & 1 \\ 1 & 7 & 4 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \\ -6 \\ 10 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

6. (a) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 6 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 3 & 0 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$

5 ea

5 ea 7. (a) $\cos^2\theta + \sin^2\theta = 1$

(b) $\begin{bmatrix} 0 & 2 & 3 & 1 \\ 4 & 6 & 6 & 1 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 5 & 1 \end{bmatrix} \to \begin{bmatrix} 4 & 6 & 6 & 1 \\ 0 & 2 & 3 & 1 \\ 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 3 \end{bmatrix}$ Two exchanges, no effect ($(-1)^2 = +1$), so $\det = 4\cdot2\cdot5\cdot3$

(c) $\begin{bmatrix} 4 & 1 & 1 \\ 2 & 0 & 4 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow[*\,1/2]{-2R_2} \begin{bmatrix} 0 & 1 & -7 \\ 1 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{-R_3} \begin{bmatrix} 0 & 0 & -8 \\ 1 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix} \to$

$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & -8 \end{bmatrix}$ $\det = \underbrace{-8}_{\text{final}} * \underbrace{(-1)^2}_{\text{two exchanges}} * \underbrace{2}_{\text{mult by } 1/2} = -16$

10 8. $\begin{bmatrix} 2 & 3 & 4 & 0 \\ 1 & 1 & -1 & 2 \\ 1 & 1 & -1 & 2 \\ 1 & 1 & -1 & 2 \end{bmatrix} \begin{matrix} \xrightarrow{-2R_2} \\ \\ \xrightarrow{-R_2} \\ \xrightarrow{-R_2} \end{matrix} \begin{array}{cccc} * & * & & \\ \hline \end{array} \begin{bmatrix} 0 & \underline{1} & 6 & -4 \\ \underline{1} & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ Basis $= \left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

5 9. $\begin{array}{cccc} * & * & * & * \end{array}$ $\begin{bmatrix} \underline{7} & 6 & 4 & 1 \\ 0 & \underline{5} & 3 & 2 \\ 0 & 0 & \underline{2} & 3 \\ 0 & 0 & 0 & \underline{4} \end{bmatrix}$ Pivots in every column, so they are lin. indep.

5 ea 10. (a) $D = \begin{bmatrix} 2 & 5 & 0 \\ 0 & & 7 \end{bmatrix}$ (b) $C = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 1 & 1 & 2 \end{bmatrix}$ (c) $A \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix} = 5 \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 15 \\ 5 \end{bmatrix}$

11.

(a) $$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 1 \\ 1 & -1 & -1 & 2 & 1 \end{array}\right] \xrightarrow[-R_1]{} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 1 \\ 0 & -2 & -2 & 2 & 0 \end{array}\right] \xrightarrow[* -1/2]{}$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & -1 & 0 \end{array}\right] \xrightarrow[-R_2]{} \left[\begin{array}{cccc|c} \underline{1} & 0 & 0 & 1 & 1 \\ 0 & \underline{1} & 1 & -1 & 0 \end{array}\right]$$

10

$$\left\{\begin{array}{l} x + w = 1 \\ y + z - w = 0 \end{array}\right\} \quad \left\{\begin{array}{l} x = 1 - w \\ y = \quad -z + w \\ z = z \\ w = w \end{array}\right\} \quad \boxed{\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + z \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} + w \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}}$$

(b) If $x = 1$ then $1 + w = 1$ so $w = 0$. So

5

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + z \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}$$

12. $$\left[\begin{array}{cccc|c} 1 & 2 & 2 & 1 & 2 \\ 2 & 4 & 4 & 2 & 4 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right] \xrightarrow[-2R_1]{-2R_3} \left[\begin{array}{cccc|c} \underline{1} & 2 & 0 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \underline{1} & 1 & 0 \end{array}\right]$$ for part (c)

5ea

(a) $$\text{Basis}(\text{Im}) = \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix} \right\}$$

(b) $x + 2y - w = 0$

$z + w = 0$

$$\text{Basis}(\text{Ker}) = \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\}$$

(c) $$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + w \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

13. (a)

$$\det\begin{bmatrix}1-\lambda & 4\\ 3 & 5-\lambda\end{bmatrix} = (1-\lambda)(5-\lambda)-12 = \lambda^2-6\lambda-7$$
$$= (\lambda-7)(\lambda+1)$$

$\boxed{\lambda=-1}$

10

$$\operatorname{Ker}\begin{bmatrix}2 & 4\\ 3 & 6\end{bmatrix} \qquad 2x+4y=0,\quad x=-2y \qquad \boxed{V=\begin{bmatrix}-2\\ 1\end{bmatrix}}$$

$\boxed{\lambda=7}$

$$\operatorname{Ker}\begin{bmatrix}-6 & 4\\ 3 & -2\end{bmatrix} \qquad 3x=2y \qquad \boxed{V=\begin{bmatrix}2\\ 3\end{bmatrix}}$$

(b)

$$\det\begin{bmatrix}2-\lambda & 1 & 1\\ 0 & 1-\lambda & 4\\ 0 & 3 & 5-\lambda\end{bmatrix} = (2-\lambda)(\text{det from (a)})$$
$$= (2-\lambda)(\lambda-7)(\lambda+1)$$

10

$\boxed{\lambda=2}$

$$\operatorname{Ker}\begin{bmatrix}0 & 1 & 1\\ 0 & -1 & 4\\ 0 & 3 & 3\end{bmatrix} \xrightarrow[-3R_1]{+R_1} \begin{array}{c} \quad\;\; * \;\; * \\ \begin{bmatrix}0 & \underline{1} & 1\\ 0 & 0 & \underline{5}\\ 0 & 0 & 0\end{bmatrix}\end{array} \qquad \boxed{V=\begin{bmatrix}1\\ 0\\ 0\end{bmatrix}}$$

$\boxed{\lambda=-1}$

$$\operatorname{Ker}\begin{bmatrix}3 & 1 & 1\\ 0 & 2 & 4\\ 0 & 3 & 6\end{bmatrix} \xrightarrow[-\frac{3}{2}R_2]{*\frac{1}{2}} \begin{bmatrix}3 & 1 & 1\\ 0 & 1 & 2\\ 0 & 0 & 0\end{bmatrix} \xrightarrow{-R_2} \begin{bmatrix}3 & 0 & -1\\ 0 & 1 & 2\\ 0 & 0 & 0\end{bmatrix} \qquad \boxed{V=\begin{bmatrix}1/3\\ -2\\ 1\end{bmatrix}}$$

$\boxed{\lambda=7}$

$$\operatorname{Ker}\begin{bmatrix}-5 & 1 & 1\\ 0 & -6 & 4\\ 0 & 3 & -2\end{bmatrix} \xrightarrow[*\frac{1}{3}]{+2R_3} \begin{bmatrix}-5 & 1 & 1\\ 0 & 0 & 0\\ 0 & 1 & \frac{-2}{3}\end{bmatrix} \xrightarrow{-R_3} \begin{bmatrix}-5 & 0 & 5/3\\ 0 & 0 & 0\\ 0 & 1 & -2/3\end{bmatrix}$$

$+5x = \frac{5}{3}z \qquad y=\frac{2}{3}z$

$x = \frac{1}{3}z$

$$\boxed{V=\begin{bmatrix}1/3\\ 2/3\\ 1\end{bmatrix} \text{ OR } \begin{bmatrix}1\\ 2\\ 3\end{bmatrix}}$$

14. (a) $\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 2 & 4 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 & 0 & 1 \end{array}\right] \begin{array}{c} \\ \xrightarrow{-R_1} \\ \xrightarrow{-R_1} \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 2 & 3 & -1 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 & 1 \end{array}\right] \xrightarrow{-2R_3}$

$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & -2 \\ 0 & 1 & 1 & -1 & 0 & 1 \end{array}\right] \begin{array}{c} \xrightarrow{-R_2} \\ \\ \xrightarrow{-R_2} \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -1 & 2 \\ 0 & 0 & 1 & 1 & 1 & -2 \\ 0 & 1 & 0 & -2 & -1 & 3 \end{array}\right]$

10

$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -1 & 2 \\ 0 & 1 & 0 & -2 & -1 & 3 \\ 0 & 0 & 1 & 1 & 1 & -2 \end{array}\right]$ where $\underbrace{\begin{bmatrix} 0 & -1 & 2 \\ -2 & -1 & 3 \\ 1 & 1 & -2 \end{bmatrix}}_{A^{-1}}$

5

(b) $\begin{bmatrix} 0 & -1 & 2 \\ -2 & -1 & 3 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -2+6 \\ -2-2+9 \\ 1+2-6 \end{bmatrix} = \boxed{\begin{bmatrix} 4 \\ 5 \\ -3 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}}$

Check: $\begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & 4 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \\ -3 \end{bmatrix} = \begin{bmatrix} 4-3 \\ 4+10-12 \\ 4+5-6 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ ok.

15 (a) Yes: distinct eigenvalues

(b) Maybe, maybe not. If $\dim \operatorname{Ker}(A-I)=2$, it is diagonalizable. If $\dim \operatorname{Ker}(A-I)=1$, it is not.

5ea

16.

(a) $V = \text{Ker}\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & -1 & 1 \end{bmatrix} \xrightarrow{-R_1} \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & -1 & -2 & 1 \end{bmatrix} \xrightarrow[*-1]{+R_2} \begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & -1 \end{bmatrix}$

$x = z - w$

$y = -2z + w$

$z = z$

$w = w$

10

$\text{Basis}(V) = \left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$

(b) $V^{\perp} = \text{Ker}\begin{bmatrix} 1 & -2 & 1 & 0 \\ -1 & 1 & 0 & 1 \end{bmatrix} \xrightarrow{+R_1} \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & -1 & 1 & 1 \end{bmatrix} \xrightarrow[*-1]{-2R_2} \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & -1 & -1 \end{bmatrix}$

$x = z + 2w$

$y = z + w$

$z = z$

$w = w$

10

$\text{Basis}(V^{\perp}) = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$

(c) $p_1 = \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \quad p_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix} - \dfrac{\begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}}{\begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}} \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix} + \frac{3}{6} \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 0 \\ 1/2 \\ 1 \end{bmatrix}$

5

$p_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \quad p_4 = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix} - \dfrac{\begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}}{\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix} - \frac{3}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$

$q_1 = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \quad q_2 = \sqrt{\frac{2}{3}} \begin{bmatrix} -1/2 \\ 0 \\ 1/2 \\ 1 \end{bmatrix}, \quad q_3 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \quad q_4 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$

5 17. $\frac{1}{\sqrt{3}}$, $\frac{-1}{\sqrt{2}}$, $\sqrt{\frac{2}{3}}\,\frac{1}{2} = \frac{1}{\sqrt{6}}$

5 18. (a)

$$\begin{array}{c|c} & \begin{array}{ccc} 1 & x & x^2 \end{array} \\ \begin{array}{c} 1 \\ x \\ x^2 \end{array} & \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \end{array}$$

$A(1) = 1 - 0$

$A(x) = x - x = 0$

$A(x^2) = x^2 - x(2x)$

10 (b) $\text{Ker}(A) = \text{Span}\{x\}$

5 (c) $\text{Im}(T) = \text{Span}\{1, x^2\}$

10 19. $V = \text{Ker}\begin{bmatrix} a & b & c & d \end{bmatrix}$ for some $\begin{bmatrix} a & b & c & d \end{bmatrix}$, so $W = \text{Ker}\begin{bmatrix} a & b & c & d \\ 1 & 0 & 1 & 0 \end{bmatrix}$. This will have either 1 or 2 pivots, so $\underline{\dim W} = \dim \ker$ = number of non-pivots is either $\underline{3 \text{ or } 2}$.