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Math 2250, Fall 2008, Final Exam

Instructions:

1. *For full credit, explain your reasoning.*
2. Quickly read the whole test before beginning work. There are 19 questions.
3. There are 240 points, distributed as indicated.
4. The following notation will be used throughout this test:

   $\mathbf{R}$ is the real numbers.

   $\mathbf{R}^n$ is the vector space of dimension n.

   P_k is the vector space of all polynomials of degree at most k.

   F is the vector space of all functions $\mathbf{R} \longrightarrow \mathbf{R}$.

Questions:

1. (5) Find the matrix for the linear transformation $A : \mathbf{R}^3 \longrightarrow \mathbf{R}^2$ such that

$$A \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \ A \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \text{ and } A \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

2. (5 each) Which of the following functions are linear transformations:

   (a) $T : F \longrightarrow \mathbf{R}^2$ by $Tf = \begin{bmatrix} f(0) \\ f(1) \end{bmatrix}$.

   (b) $T : \mathbf{R}^3 \longrightarrow P_2$ by $T \begin{bmatrix} a \\ b \\ c \end{bmatrix} = ax^2 + bx + c$.

   (c) $T : \mathbf{R}^2 \longrightarrow \mathbf{R}^2$ by $T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x - 1 \\ y \end{bmatrix}$.

3. (5 each) Which of the following subsets are subspaces?

   (a) $\left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbf{R}^3 \mid x + y = 1 \right\}$.

   (b) $\left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbf{R}^3 \mid x + y = z \right\}$.

   (c) $\{f \in F \mid f(1) = 2f(0)\}$.

4. (5) Write in matrix form:

$$\begin{aligned} 3x + 2y + z &= 1 \\ 5x - y - z &= 0 \\ 4x + 4y + z &= 3 \end{aligned}$$

5. (5) Write the following vector equation as a matrix equation of the form $Ax = b$:

$$2\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + 5\begin{bmatrix} 3 \\ 7 \\ 7 \end{bmatrix} - 6\begin{bmatrix} 0 \\ 2 \\ 4 \end{bmatrix} + 10\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

6. (5 each) Give the 4 by 4 elementary matrices which accomplish the following elementary row operations:

   (a) Multiply row 4 by 6.

   (b) Add 3 times row 2 to row 4.

   (c) Exchange rows 1 and 4.

7. (5 each) Find the determinants of the following matrices:

   (a) $\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$

   (b) $\begin{bmatrix} 0 & 2 & 3 & 1 \\ 4 & 6 & 6 & 1 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 5 & 1 \end{bmatrix}$

   (c) $\begin{bmatrix} 4 & 1 & 1 \\ 2 & 0 & 4 \\ 0 & 1 & 1 \end{bmatrix}$

8. (10) Find a basis for the span of

$$\left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 2 \\ 2 \end{bmatrix} \right\}.$$

9. (5) Are the vectors

$$\left\{ \begin{bmatrix} 7 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 6 \\ 5 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \right\}.$$

linearly independent? Why or why not?

10. (15) Suppose that A has eigenvalues $\lambda_1 = 2$, $\lambda_2 = 5$ and $\lambda_3 = 7$, with corresponding eigenvectors $v_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$, $v_2 = \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}$ and $v_3 = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$.

    (a) Find a diagonal form D of A.

    (b) Find a matrix C such that $A = CDC^{-1}$.

    (c) Compute $A \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}$.

11. (a) (10) Find all solutions $\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$ to

$$\begin{aligned} x + y + z &= 1 \\ x - y - z + 2w &= 1 \end{aligned}$$

    (b) (5) Find all the solutions which also satisfy $x = 1$.

12. (5 each) Let $T : \mathbf{R}^4 \longrightarrow \mathbf{R}^3$ be

$$\begin{bmatrix} 1 & 2 & 2 & 1 \\ 2 & 4 & 4 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

(a) Find a basis for $\mathrm{Im}(T)$.

(b) Find a basis for $\mathrm{Ker}(T)$.

(c) Find all solutions to $Tv = \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix}$.

13. (10 each) Find the eigenvalues and eigenvectors of

(a) $\begin{bmatrix} 1 & 4 \\ 3 & 5 \end{bmatrix}$.

(b) $\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 4 \\ 0 & 3 & 5 \end{bmatrix}$.

14. (a) (10) Find $\begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & 4 \\ 1 & 1 & 2 \end{bmatrix}^{-1}$.

(b) (5) Solve $\begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & 4 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ without doing any more row reduction.

15. (5 each)

(a) Suppose that the characteristic polynomial of A is $(\lambda - 1)(\lambda - 2)(\lambda - 4)$. Is it possible to diagonalize A?

(b) Suppose that the characteristic polynomial of A is $(\lambda - 1)^2(\lambda - 2)(\lambda - 4)$. Is it possible to diagonalize A?

16. (25) Let V be the subspace of $\mathbf{R}^4$ containing those $\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$ such that $x+y+z=0$ and $x-z+w=0$.

(a) Find a basis for V.

(b) Find a basis for $V^{\perp}$.

(c) Use the Gram-Schmidt procedure to orthonormalize the basis for $\mathbf{R}^4$ obtained by concatenating the bases you found for V and $V^{\perp}$.

17. (5) Find the coordinates of $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ with respect to the orthonormal basis

$$\left\{ \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \frac{\sqrt{2}}{\sqrt{3}} \begin{bmatrix} 1/2 \\ 1/2 \\ -1 \end{bmatrix} \right\}$$

18. Let $A : P_2 \longrightarrow P_2$ be the linear transformation $A(p) = p - xp'$.

(a) (5) Find the matrix for A with respect to the basis $\{1, x, x^2\}$ for P_2.

(b) (10) Compute $\mathrm{Ker}(T)$.

(c) (5) Compute $\mathrm{Im}(T)$.

19. (10) Suppose that V is a 3-dimensional subspace of $\mathbf{R}^4$. If we let W be the subset containing the vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ in V such that $x_1 + x_3 = 0$, what are the possibilities for the dimension of W?

– The End –