Solutions to Sample Final from previous Math 2150

5 1. (a) $y' + y = 0$

5 (b) $y' + (y)^2 = 0$

5 (d) $y'' + y' + y = 0$

5 2. $\boxed{\text{No,}}$ if more than 1 solution, then ∞ many.

10 3. $\begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 3 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ so $\boxed{x = 2x_2 + x_1} = \begin{pmatrix} 0 \\ 2 \\ -2 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \boxed{\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}}$

solves $Ax = \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix}$.

10 4. $L[2y_p] = 2g$ so $y = 2y_p + c_1 y_1 + c_2 y_2$. Now

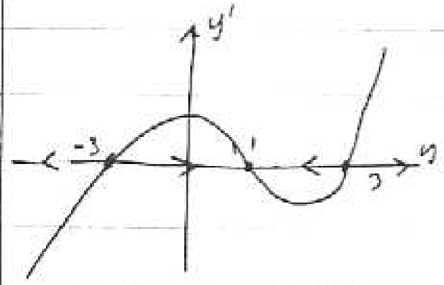
$$0 = 2y_p(0) + c_1 y_1(0) + c_2 y_2(0)$$
$$0 = 2y_p'(0) + c_1 y_1'(0) + c_2 y_2'(0)$$

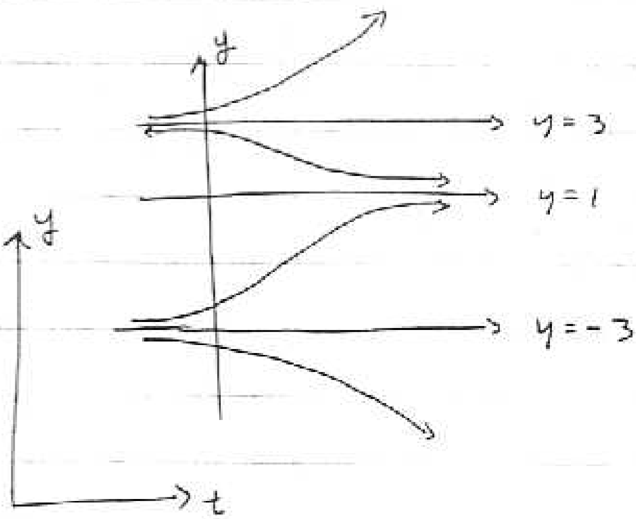
OR

$$0 = 4 + c_1 + 0$$
$$0 = 2 + 0 + c_2$$

so $\boxed{y = 2y_p - 4y_1 - 2y_2}$

5.





2 graph y'

2 0's $\rightsquigarrow$ equil sol's

2 pos $\rightsquigarrow$ incr

2 neg $\rightsquigarrow$ decr

10 2 asympt to eq. sol

6 (a) $y' - 2xy = 0$ OR $\frac{y'}{y} = 2x$

$p = -2x$

$\mu = \exp \int p = e^{-x^2}$ $\ln|y| = x^2$

10 $y = c/\mu = Ce^{x^2}$ $y = Ce^{x^2}$

(b) $y' - 2xy = x$

$\mu = e^{-x^2}$ as above

$y = e^{x^2} \int x e^{-x^2} dx$ $\int x e^{-x^2} dx = \frac{1}{2} \int e^{-u} du = -\frac{1}{2} e^{-u}$

$u = x^2$ $= -\frac{1}{2} e^{-x^2}$

$\frac{1}{2} du = x\,dx$

so

5 $y = -\frac{1}{2} + Ce^{x^2}$

(c) $y(0) = 1$ gives $1 = -\frac{1}{2} + Ce^{0}$ or $\frac{3}{2} = C$

so $y = -\frac{1}{2} + \frac{3}{2} e^{x^2}$

5

(1) $\frac{dy}{1+2y} = x$

$\frac{1}{2} \ln|1+2y| = \frac{1}{2}x^2 + C$

$1+2y = Ae^{x^2}$

$y = -\frac{1}{2} + Be^{x^2}$

7 (a) $y'' - 6y' + 10y = 0$

$r^2 - 6r + 10 = (r-3)^2 + 1 = 0$ $\boxed{r = 3 \pm i}$

10 $\boxed{y = c_1 e^{3t}\cos t + c_2 e^{3t}\sin t}$

(b) $y'' - 6y' + 10y = x + e^{2x}$

x: $y_p = ax + b$ e^{2x}: $y_p = Ae^{2x}$

$y_p' = a$ $y_p' = 2y_p$, $y_p'' = 4y_p$

$y_p'' = 0$ $y_p'' - 6y_p' + 10y_p = (4 - 12 + 10)Ae^{2x}$

$-6a + 10ax + 10b = x$ $= 2Ae^{2x}$

$a = \frac{1}{10}$, $10b - \frac{6}{10} = 0$ $y_p = \frac{1}{2}e^{2x}$

$10b = \frac{6}{10}$, $b = \frac{6}{100}$

5

$$y = \boxed{\frac{1}{10}x + \frac{6}{100}} + \boxed{\frac{1}{2}e^{2x}} + c_1 e^{3x}\cos x + c_2 e^{3x}\sin x$$

5

(c) $y'''' - 16y = 0$ $\boxed{y = c_1 e^{2x} + c_2 e^{-2x} + c_3 \sin 2x + c_4 \cos 2x}$

$r^4 - 16 = 0$ OR

10 $r^2 = \pm 4$ $\boxed{y = c_1 \cosh 2x + c_2 \sinh 2x + c_3 \cos 2x + c_4 \sin 2x}$

$\boxed{r = \pm 2, \pm 2i}$

8.

10

$$e^{At} = \begin{bmatrix} e^{2t} & & & \\ & e^{2t} & & \\ & & e^{2t} & \\ & & & e^{7t} \end{bmatrix} \begin{bmatrix} 1 & t & t^2/2 & 0 \\ & 1 & t & 0 \\ & & 1 & 0 \\ & & & 1 \end{bmatrix} = \begin{bmatrix} e^{2t} & te^{2t} & \frac{t^2}{2}e^{2t} & 0 \\ 0 & e^{2t} & te^{2t} & 0 \\ 0 & 0 & e^{2t} & 0 \\ 0 & 0 & 0 & e^{7t} \end{bmatrix}$$

9.
$$\left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 2 & 4 \\ 2 & 2 & 0 & 4 & 0 & 6 \\ 1 & 1 & 2 & 2 & 6 & 7 \\ 0 & 0 & 1 & 5 & 8 & 7 \end{array}\right] \rightsquigarrow \begin{array}{c} \\ -2\text{①} \\ -\text{①} \\ -\text{new ③} \end{array} \left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 2 & 4 \\ 0 & 0 & -2 & 2 & -4 & -2 \\ 0 & 0 & 1 & 1 & 4 & 3 \\ 0 & 0 & 0 & 4 & 4 & 4 \end{array}\right]$$

$$\begin{array}{c} \\ \to \text{③} \\ \tfrac{1}{4}\text{④} \\ \text{②}+2\text{③} \end{array} \left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 2 & 4 \\ 0 & 0 & 1 & 1 & 4 & 3 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 4 & 4 & 4 \end{array}\right] \rightsquigarrow \begin{array}{c} -\text{②} \\ \\ \\ \\ \end{array} \left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & -2 & 1 \\ 0 & 0 & 1 & 0 & 3 & 2 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & & & & & 0 \end{array}\right]$$

10

$x_1 = 1 - x_2 + 2x_5$
$x_3 = 2 - 3x_5$
$x_4 = 1 - x_5$
x_2, x_5 arbitrary

5 Dim(solⁿ space) = 2

5 Rank = 3 (= 5 − 2)

10. $(2-\lambda)(3-\lambda) - 2 = \lambda^2 - 5\lambda + 4 = (\lambda-1)(\lambda-4)$

$A = \begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix}$ $\quad A - 1 = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\quad \boxed{v = \begin{bmatrix} 1 \\ -1 \end{bmatrix}}\ \boxed{\lambda = 1}$

20 (5 ea)

$A - 4 \quad \begin{pmatrix} -2 & 1 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\quad \boxed{v = \begin{bmatrix} 1 \\ 2 \end{bmatrix}}\ \boxed{\lambda = 4}$

11.

10 $\quad y = c_1 e^{t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 e^{4t}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

5

12. (a) $x_1 = y$ $\quad x_1' = y' = x_2$

$x_2 = y'$ $\quad x_2' = y'' = 5y - 6y' = 5x_1 - 6x_2$

$$\boxed{X' = \begin{bmatrix} 0 & 1 \\ 5 & -6 \end{bmatrix} X}$$

5

(b) $x_1 = y$ $\quad x_1' = x_2$

$x_2 = y'$ $\quad x_2' = x_3$

$x_3 = y''$ $\quad x_3' = y - 10y' + y''$

$$\boxed{X' = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -10 & 1 \end{bmatrix} X}$$

13. (a) + (b) $\mathcal{L}\{y'\} = sY$, $\mathcal{L}\{y''\} = s^2Y - 1$

5

(a) $s^2Y - 1 - 5sY + 6Y = 0$ $\quad \boxed{Y = \frac{1}{s^2-5s+6} = \frac{1}{(s-2)(s-3)}}$

$= \frac{a}{s-2} + \frac{b}{s-3}$ $\quad 1 = a(s-3) + b(s-2) = (a+b)s - (3a+2b)$

$b = -a$, $3a - 2a = -1$, $a = -1, b = 1$

5

$$\boxed{y = -e^{2t} + e^{3t}}$$

(b) $s^2Y - 1 + 25Y = \frac{1}{s+3}$

$(s^2+25)Y = 1 + \frac{1}{s+3}$

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$$\boxed{Y = \frac{1}{s^2+25} + \frac{1}{(s+3)(s^2+25)}} = \frac{s+4}{(s^2+25)(s+3)} = \frac{a}{s+3} + \frac{bs+c}{s^2+25}$$

$s + 4 = as^2 + 25a + bs^2 + 3bs + cs + 3c = (a+b)s^2 + (3b+c)s + (25a+3c)$

$a + b = 0$ $\quad 3b + c = 1$ $\quad 25a + 3 + 9a = 4$ $\quad a = \frac{1}{34}, b = -\frac{1}{34}, c = \frac{37}{34}$

$b = -a$ $\quad c = 1 + 3a$ $\quad 34a = 1$

$$Y = \frac{1}{34}\left(\frac{1}{s+3} - \frac{s}{s^2+25} + \frac{37}{5}\,\frac{5}{s^2+25}\right)$$

5

$$\boxed{y = \frac{1}{34}\left(e^{-3t} - \cos(5t) + \frac{37}{5}\sin(5t)\right)}$$

(c) $s^2 Y - 1 + 25Y = 5e^{-\pi s}$

$$(s^2+25)Y = 1 + 5e^{-\pi s}$$

$$Y = \frac{1}{s^2+25} + e^{-\pi s}\frac{5}{s^2+25}$$

$$y = \frac{1}{5}\sin(5t) + u(t-\pi)\sin(5(t-\pi))$$

(sufficient) $\rightarrow$ $= \frac{1}{5}\sin(5t) - u(t-\pi)\sin(5t)$

$$= \begin{cases} \frac{1}{5}\sin(5t) & t < \pi \\ -\frac{4}{5}\sin(5t) & t > \pi \end{cases}$$

(d) $s^2 Y - 1 + 25Y = \frac{1}{s} - e^{-\pi s}\frac{1}{s}$

$$(s^2+25)Y = 1 + \frac{1}{s} - e^{-\pi s}\frac{1}{s}$$

$$Y = \frac{1}{s^2+25} + \frac{1}{s(s^2+25)}\left(1 - e^{-\pi s}\right)$$

$$\frac{1}{s(s^2+25)} = \frac{a}{s} + b\frac{s}{s^2+25} + c\frac{5}{s^2+25}$$

$$1 = a(s^2+25) + bs(s) + c5(s)$$

$$1 = (a+b)s^2 + 5cs + 25a$$

comparing coefficients of s^2, s and 1 we get

$a+b=0$ $\quad b = -1/25$

$5c = 0$ $\quad c = 0$

$25a = 1$ $\quad a = 1/25$

$$\frac{1}{s(s^2+25)} = \frac{1}{25}\left(\frac{1}{s}\right) - \frac{1}{25}\frac{s}{s^2+25}$$

$$Y = \frac{1}{s^2+25} + \left(\frac{1}{25}\right)\left(\frac{1}{s} - \frac{s}{s^2+25}\right)\left(1 - e^{-\pi s}\right)$$

$$y = \frac{1}{5}\sin(5t) + \frac{1}{25}\left(1 - \cos(5t)\right) - \frac{1}{25}\left(u(t-\pi)\right)\left(1 - \cos(5(t-\pi))\right)$$

(sufficient) $\rightarrow$ $= \frac{1}{25} + \frac{1}{5}\sin(5t) - \frac{1}{25}\cos(5t) - \frac{1}{25}u(t-\pi)\left(1 + \cos(5t)\right)$

$$= \begin{cases} \frac{1}{25} + \frac{1}{5}\sin(5t) - \frac{1}{25}\cos(5t) & t < \pi \\ \frac{1}{5}\sin(5t) - \frac{2}{25}\cos(5t) & t > \pi \end{cases}$$