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Use Laplace transforms to solve the differential equation

$$y'' + y = \begin{cases} 1 & t < \pi \\ 0 & t > \pi \end{cases}, \qquad y(0) = 0, \qquad y'(0) = 0.$$

Let $Y = \mathcal{L}\{y\}$. Then $\mathcal{L}\{y'\} = sY$ and $\mathcal{L}\{y''\} = s^2 Y$.

The right hand side

$$\begin{cases} 1 & t < \pi \\ 0 & t > \pi \end{cases} = 1 - u(t-\pi)$$

So the transformed equation is

$$(s^2+1)\,Y = \frac{1}{s} - e^{-\pi s}\frac{1}{s} = \left(1 - e^{-\pi s}\right)\frac{1}{s}$$

with solution $Y = \left(1 - e^{-\pi s}\right)\dfrac{1}{s(s^2+1)}$

Now $$\frac{1}{s(s^2+1)} = \frac{A}{s} + \frac{Bs}{s^2+1} + \frac{C}{s^2+1} \underset{\text{usual method}}{=} \frac{1}{s} - \frac{s}{s^2+1}$$

so $$\boxed{y = 1 - \cos t - u(t-\pi)\left[1 - \cos(t-\pi)\right]}$$

sufficient answer, but some simplification is possible

$$y = 1 - \cos t - u(t-\pi)\left[1 + \cos t\right]$$

$$y = \begin{cases} 1 - \cos t & t < \pi \\ -2\cos t & t > \pi \end{cases}$$