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Math 2150, Fall 2005, Quiz 12
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The eigenvalues of $\begin{bmatrix} 3 & 0 \\ -1 & 3 \end{bmatrix}$ are both 3. (You do not need to check this.)
Find the general solution to the differential equation

$$y' = \begin{bmatrix} 3 & 0 \\ -1 & 3 \end{bmatrix} y.$$

(Partial credit: find at least one solution.)

First, eigenvector(s): $A - 3I = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}$ so $(A-3I)\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ iff $x=0$.

Eigenvector (only one) $v = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. $\boxed{\text{Solution } y = e^{3t}\begin{bmatrix} 0 \\ 1 \end{bmatrix}.}$

Second, generalized eigenvector: $(A-3I)\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ leads to the problem: $\left[\begin{array}{cc|c} 0 & 0 & 0 \\ -1 & 0 & 1 \end{array}\right]$ so $x=-1$, y arbitrary. Let $v_2 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$.

$$\boxed{\text{Second solution: } y = e^{3t}\begin{bmatrix} -1 \\ 0 \end{bmatrix} + te^{3t}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = e^{3t}\begin{bmatrix} -1 \\ t \end{bmatrix}}$$

$$\boxed{\begin{aligned} \underline{\text{General solution}}: \quad y &= c_1 e^{3t}\begin{bmatrix} 0 \\ 1 \end{bmatrix} + c_2 e^{3t}\begin{bmatrix} -1 \\ t \end{bmatrix} \\ &= e^{3t}\begin{bmatrix} -c_2 \\ c_1 + c_2 t \end{bmatrix} \end{aligned}}$$

Alternate method, matrix exponential: $A - 3I = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}$ so $(A-3I)^2 = 0$ and $e^{At} = e^{3t}(I + (A-3I)t) = e^{3t}\begin{bmatrix} 1 & 0 \\ -t & 1 \end{bmatrix}$, giving

$y_1 = e^{3t}\begin{bmatrix} 1 \\ -t \end{bmatrix}$, $y_2 = e^{3t}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and general solution $c_1 y_1 + c_2 y_2$.

Note that this produces the same 2-dimensional family of functions, as it must.