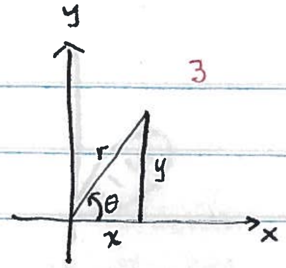
$\boxed{\text{Test 3}}$ $\boxed{\text{math 2030}}$ $\boxed{\text{W '16}}$

1. 

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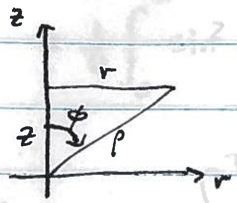
(a) $x = r\cos\theta$ [2]

$y = r\sin\theta$

(b) $dA = r\,dr\,d\theta$ [3]

(c) $dV = r\,dr\,d\theta\,dz$ [2]

2. 

(a) $r = \rho\sin\phi$

$z = \rho\cos\phi$

(b) $dr\,dz = \rho\,d\rho\,d\phi$

(c) $dV = \rho^2\sin\phi\,d\rho\,d\phi\,d\theta$

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3. $\iint_R 1\,dA \le \iint_R f\,dA \le \iint_R 4-x-y\,dA$

$\iint_R 1\,dA = (2-0)(1-0) = 2$

$\iint_R 4-x-y\,dA = \int_0^2\int_0^1 4-x-y\,dy\,dx$

$= \int_0^2 4y-xy-\tfrac12 y^2\Big|_0^1 = \int_0^2 4-x-\tfrac12\,dx$

$= 4x-\tfrac12 x^2-\tfrac12 x\Big|_0^2 = 8-\tfrac42-\tfrac22 = 5$

$\boxed{\text{It is between 2 and 5.}}$

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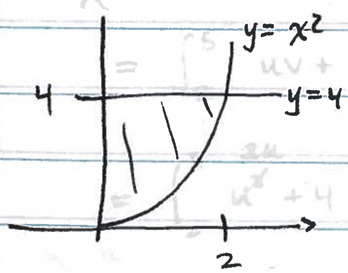
4. 

$\boxed{\text{Area}} = \int_0^2\int_{x^2}^4 dy\,dx = \int_0^2 4-x^2\,dx$

$= 4x-\frac{x^3}{3}\Big|_0^2 = 8-\frac83 = \boxed{\frac{16}{3}}$

Tot $x = \int_0^2 x(4-x^2)\,dx = 2x^2-\frac{x^4}{4}\Big|_0^2 = 8-4 = 4$ so $\boxed{\bar{x} = \frac{4}{16/3} = \frac{3}{4}}$

Tot $y = \int_0^2\int_{x^2}^4 y\,dy\,dx = \frac12\int_0^2 16-x^4\,dx = \frac12\left(16x-\frac{x^5}{5}\Big|_0^2\right)$

$= 16-\frac{16}{5} = \frac{16\cdot4}{5}$ so $\boxed{\bar{y} = \frac{16\cdot4}{5}\cdot\frac{3}{16} = \frac{12}{5}}$

Centroid $= \boxed{\left(\frac43, \frac{12}{5}\right)}$

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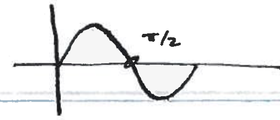
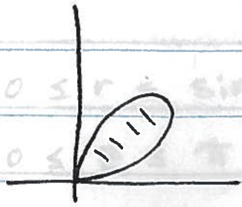
$r = \sin 2\theta$



(5)



$\boxed{\text{Area}} = \int_0^{\pi/2}\int_0^{\sin 2\theta} r\,dr\,d\theta = \int_0^{\pi/2} \frac{1}{2}\sin^2 2\theta \, d\theta$

$u = 2\theta$
$du = 2d\theta$

$= \frac{1}{4}\int_0^{\pi} \sin^2 u \, du = \boxed{\frac{\pi}{8}}$

WRONG. $\int r^2 dr = \frac{1}{3}r^3$, not $\frac{1}{2}r^2$

ERROR see 2 pages later.

Tot x : $\int_0^{\pi/2}\int_0^{\sin(2\theta)} r\cos\theta \, r\,dr\,d\theta = \cancel{\int_0^{\pi/2} \cos\theta \cdot \frac{1}{2}\sin^2 2\theta \, d\theta}$

$= \cancel{2\int_0^{\pi/2} \sin^2\theta\cos^3\theta\,d\theta} = \cancel{2\int_0^{\pi/2} (\sin^2\theta - \sin^4\theta)\cos\theta\,d\theta}$

$\cancel{u = \sin\theta}$
$du = \cancel{\cos\theta\,d\theta}$

$= \cancel{2\int_0^1 u^2 - u^4\,du} = \cancel{2\left(\frac{1}{3} - \frac{1}{5}\right)} = \cancel{\frac{1}{15}}$

So $\boxed{\cancel{\bar{x} = \frac{1}{15}\cdot\frac{8}{\pi} = \frac{8}{15\pi}}}$. By symmetry $\boxed{\bar{y} = \bar{x}.}$ $\boxed{= \frac{128}{105\pi}}$

(6) $\iint_R dA = \int_2^5\int_1^3 \frac{\partial(x,y)}{\partial(u,v)}\,dv\,du = \int_2^5\int_1^3 u + v\,dv\,du$

$= \int_2^5 uv + \frac{1}{2}v^2\Big|_{v=1}^{v=3} du = \int_2^5 3u + \frac{9}{2} - \left(u + \frac{1}{2}\right) du$

$= \int_2^5 \overset{2u}{u^2} + 4\,du = \cancel{\frac{1}{3}u^3 + 2u^2\Big|_2^5 = \frac{125}{3} + 50 - \left(\frac{8}{3} + 8\right)}$

$= \cancel{\frac{117}{3} + 42 = 39 + 42 = \boxed{71}}$ $\quad \cancel{= \frac{25}{2} + 50 - 10}$

$\cancel{= \frac{25}{2} + 40} = \boxed{\cancel{\frac{105}{2}}}$

$\to = u^2 + 4u\Big|_2^5$

$= 25 + 20 - (4 + 8) = 45 - 12 = 33$

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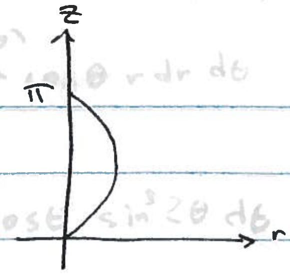
7. $0 \le r \le \sin z$

$0 \le z \le \pi$



$$\text{Vol} = \int_0^{2\pi}\int_0^{\pi}\int_0^{\sin z} r\,dr\,dz\,d\theta = 2\pi\int_0^{\pi}\tfrac{1}{2}\sin^2 z\,dz = \pi\cdot\frac{\pi}{2} = \boxed{\frac{\pi^2}{2}}$$

8.

$$I_z = \int_0^{2\pi}\int_0^{\pi}\int_0^{\sin z} r^2\cdot r\,dr\,dz\,d\theta = 2\pi\int_0^{\pi}\tfrac{1}{4}\sin^4 z\,dz$$

$$= \frac{\pi}{2}\cdot\frac{3\pi}{8} = \boxed{\frac{3\pi^2}{16}} = \text{Vol} * \frac{3}{8}$$

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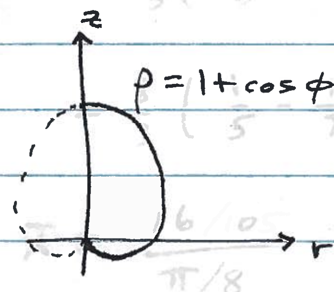
9.



$$\text{Vol} = \int_0^{2\pi}\int_0^{\pi}\int_0^{1+\cos\phi}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

$$= 2\pi\int_0^{\pi}\tfrac{1}{3}(1+\cos\phi)^3\sin\phi\,d\phi$$

$u = 1+\cos\phi \qquad u(0) = 2$

$-du = \sin\phi\,d\phi \qquad u(\pi) = 0$

$$= \frac{2\pi}{3}\int_0^2 u^3\,du = \frac{\pi}{6}u^4\Big|_0^2 = \boxed{\frac{8\pi}{3}}$$

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10.

$$z = \rho\cos\phi$$

$$\text{Tot } z = \int_0^{2\pi}\int_0^{\pi}\int_0^{1+\cos\phi}\rho\,\cancel{\sin\phi}\cos\phi\cdot\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

$$\cancel{= 2\pi\int_0^{\pi}\tfrac{1}{4}(1+\cos\phi)^4\sin^2\phi\,d\phi} = 2\pi\int_0^{\pi}\tfrac{1}{4}(1+\cos\phi)^4\cos\phi\sin\phi\,d\phi$$

Same substitution as in #9. Note $\cos\phi = 1-u$.

$$= \frac{\pi}{2}\int_0^2 u^4(1-u)\,du = \frac{\pi}{2}\int_0^2 u^4-u^5\,du$$

order reversed.

Error : $u-1$!

$$= \frac{\pi}{2}\left(\tfrac{1}{5}u^5 - \tfrac{1}{6}u^6\right) = \frac{\pi}{2}\left(\frac{32}{5}-\frac{32}{3}\right) = 16\pi\cdot\frac{2}{15} = \frac{32\pi}{15}$$

So $\bar{z} = \dfrac{32\pi}{15}\cdot\dfrac{3}{8\pi} = \boxed{\dfrac{4}{5}}$

5. $\bar{x}: \int_0^{\pi/2}\int_0^{\sin(2\theta)} r\cos\theta\, r\,dr\,d\theta = \int_0^{\pi/2}\cos\theta \int_0^{\sin 2\theta} r^2\,dr\,d\theta$

$$= \frac{1}{3}\int_0^{\pi/2}\cos\theta\,\sin^3 2\theta\,d\theta = \frac{8}{3}\int_0^{\pi/2}\sin^3\theta\cos^4\theta\,d\theta$$

$$(\sin 2\theta = 2\sin\theta\cos\theta)$$

$$= \frac{8}{3}\int_0^{\pi/2}(1-\cos^2\theta)\cos^4\theta\,(\sin\theta\,d\theta)$$

$u = -\cos\theta$
$du = \sin\theta\,d\theta$

$u(0) = -1$
$u(\pi/2) = 0$

$$= \frac{8}{3}\int_{-1}^{0} u^4 - u^6\,du$$

$$= \frac{8}{3}\left(\frac{u^5}{5} - \frac{u^7}{7}\Big|_{-1}^{0}\right) = \frac{8}{3}\left(0 - \left(\frac{-1}{5} - \frac{-1}{7}\right)\right)$$

$$= \frac{8}{3}\left(\frac{1}{5} - \frac{1}{7}\right) = \frac{8}{3}\left(\frac{2}{35}\right) = \frac{16}{105}$$

$$\bar{x} = \frac{16/105}{\pi/8} = \frac{16}{105}\cdot\frac{8}{\pi} = \boxed{\frac{128}{105\pi}}$$

6. Other order: $\int_1^3\int_2^5 u+v\,du\,dv = \int_1^3 \frac{1}{2}u^2 + uv\Big|_{u=2}^{u=5} dv = \int_1^3 \frac{25}{2} + 5v - (2+2v)\,dv$

$$= \int_1^3 \frac{21}{2} + 3v\,dv = \frac{21}{2}v + \frac{3}{2}v^2\Big|_1^3 = \frac{63}{2} + \frac{27}{2} - \left(\frac{21}{2} + \frac{3}{2}\right)$$

$$= 45 - 12 = 33$$