Solutions to Test 2, M 2030, W16

1. Let $f = x^2y + xy^3$. Then

$$f_x = 2xy + y^3 \qquad f_{xx} = 2y$$
$$f_y = x^2 + 3xy^2 \qquad f_{xy} = 2x + 3y^2$$
$$f_{yy} = 6xy$$

2. $f(x,y) = x^2 + x^2y + y^2$

(a) $z = f(x,y)$ is $f(1,2) = 1^2 + 1^2 \cdot 2 + 2^2 = 1 + 2 + 4 = 7$ at $(1,2)$.
$\nabla f = (f_x, f_y) = (2x + 2xy, x^2 + 2y) = (2+4, 1+4) = (6,5)$ at $(1,2)$. So the tangent plane is

$$z - 7 = 6(x-1) + 5(y-2)$$

(b) f increases most rapidly in the direction of $\nabla f = (6,5)$

(c) f neither increases nor decreases in the perpendicular direction $(-5,6)$

3. $d(x^2yz + xy^2z + xyz^2)$

$$= (2xyz + y^2z + yz^2)\,dx + (x^2z + 2xyz + xz^2)\,dy + (x^2y + xy^2 + 2xyz)\,dz$$
$$= (2\cdot1\cdot2\cdot3 + 2^2\cdot3 + 2\cdot3^2)\,dx + (1^2\cdot3 + 2\cdot1\cdot2\cdot3 + 1\cdot3^2)\,dy + (1^2\cdot2 + 1\cdot2^2 + 2\cdot1\cdot2\cdot3)\,dz$$
$$= (12 + 12 + 18)\,dx + (3 + 12 + 9)\,dy + (2 + 4 + 12)\,dz$$
$$= 42\,dx + 24\,dy + 18\,dz \qquad \text{at } (1,2,3)$$

So the tangent plane there is

(a) $$42(x-1) + 24(y-2) + 18(z-3) = 0$$

OR

$$7(x-1) + 4(y-2) + 3(z-3) = 0$$

3. (cont.)

(b) $\nabla f = \overset{6(7,4,3)}{(42, 24, 18)}$ is normal to the curve and $(1,1,-1)$ is also, so their cross-product ~~is~~ is tangent to the curve:

$$\begin{vmatrix} i & j & k \\ 7 & 4 & 3 \\ 1 & 1 & -1 \end{vmatrix} = (-7, -(-7-3), 7-4) = (-7, 10, 3)$$

4. $$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} = 2\frac{\partial f}{\partial x} + 3\frac{\partial f}{\partial y}$$ $\quad x = 2s - t$, $y = 3s + 4t$

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} = -\frac{\partial f}{\partial x} + 4\frac{\partial f}{\partial y}$$

5. $C = x^2y + xy^3$

(a) $dC = (2xy + y^3)dx + (x^2 + 3xy^2)dy$

(b) $C(2.99, 2.01) \simeq C(3,2) + dC$

$$= 42 + (2\cdot 3\cdot 2 + 2^3)(-.01) + (3^2 + 3\cdot 3\cdot 4)(.01)$$

$$= 42 + 20(-.01) + (45)(.01)$$

$$= 42 + .25 = 42.25$$

6. $z(x,y)$ defined by $xyz - x - y^3 - z^3 = 0$. Then

$\frac{\partial}{\partial x}$ gives $yz + xy\frac{\partial z}{\partial x} - 1 - 3z^2\frac{\partial z}{\partial x} = 0$

so $(yz - 1) + (xy - 3z^2)\frac{\partial z}{\partial x} = 0$ and $\frac{\partial z}{\partial x} = \frac{1 - yz}{xy - 3z^2}$.

$\frac{\partial}{\partial y}$ gives $xz + xy\frac{\partial z}{\partial y} - 3y^2 - 3z^2\frac{\partial z}{\partial x} = 0$ so, similarly,

$$\frac{\partial z}{\partial y} = \frac{3y^2 - xz}{xy - 3z^2}$$

7. $f(x,y) = x^4 - 2x^2 + y^2 - 2y$.

Find & classify critical points.

$f_x = 4x^3 - 4x = 4x(x^2-1)$ so $f_x = 0 \iff x = -1, 0,$ or 1

$f_y = 2y - 2$ so $f_y = 0 \iff y = 1$

	$(-1,0)$	$(0,0)$	$(1,0)$
$f_{xx} = 12x^2 - 4$	8	-4	8
$f_{xy} = 0$	0	0	0
$f_{yy} = 2$	2	2	2
D	16	-8	16
	min ($f_{xx} > 0$)	saddle	min

8. $L = x^2 + y^3 - \lambda(x^2 + y^2 - 1)$

$L_x = 2x - 2x\lambda = 2x(1-\lambda)$ so $x = 0$ (and $y = \pm 1$) OR $\lambda = 1$

$L_y = 3y^2 - 2y\lambda = y(3y - 2\lambda)$ so $y = 0$ (and $x = \pm 1$) OR $y = \frac{2}{3}\lambda$

$L_x = 0$

$L_y = 0$		$x = 0$, $y = \pm 1$	$\lambda = 1$
	$y = 0$, $x = \pm 1$	X	$(1,0)$ $x^2 + y^3 = 1$ $(-1,0)$ $x^2 + y^3 = 1$
	$y = \frac{2}{3}\lambda$	$(0,1)$ $x^2 + y^3 = 1$ $(0,-1)$ $x^2 + y^3 = -1$	$y = \frac{2}{3}$, $x^2 = 1 - \frac{4}{9} = \frac{5}{9}$ $x^2 + y^3 = \frac{5}{9} + \frac{8}{27} = \frac{23}{27}$

Max $= 1$ at $(\pm 1, 0)$, and $(0,1)$.

Min $= -1$ at $(0,-1)$