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Math 2030, Winter 2004, Final Exam
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1. (5) Draw the diagram which shows the relation between (x, y) and (r, θ). Write the formulas for x and y in terms of r and θ, and for $dA = dx\,dy$ in terms of r and θ.

2. (5) Draw the diagram which shows the relation between (r, z) and (ρ, ϕ). Write the formulas for r and z in terms of ρ and ϕ, and for $dA = dr\,dz$ in terms of ρ and ϕ.

3. (5) The mapping $(x, y) = F(u, v)$ is very complicated, but its Jacobian is always between 3/4 and 1. If R is a 3 by 4 rectangle, what can you say about the area of $F(R)$?

4. (5) Is (x, x^2, z) a conservative vector field? Why or why not?

5. (10) Compute $\int_C \nabla f \cdot d\vec{r}$ where $f(x, y) = x + y^2$ and C is the curve which follows a straight line from (2,0) to (1,1), then the parabola $y = x^2$ to (2,4), and finally, a straight line from (2,4) to (1,3).

6. Let $C(t) = (t^2 - t, t - t^3)$ for $0 \le t \le 1$.
   (a) (10) Compute $\int_C x dy$.
   (b) (10) Use Green's Theorem to rewrite $\int_C x dy$ as an integral over the region R inside the loop which C forms in the second quadrant.
   (c) (5) What is the area of R?

7. (10) Decompose $(4, 2, 5)$ as the sum of a vector parallel to $(1, 2, 2)$ and a vector perpendicular to $(1, 2, 2)$.

8. (10) Find the point of $x + 4y + 9z = 14$ closest to the origin.

9. (5 ea.) Let $\vec{r}(t) = (t^2 - t, t^2 + t, t^3 - t)$.
   (a) Find the velocity $\vec{v} = \vec{r}\,'$.
   (b) Find the acceleration $\vec{a} = \vec{r}\,''$.
   (c) Find the speed at time $t = 0$.
   (d) Find the normal and tangential components of acceleration at time $t = 0$.
   (e) Find the cosine of the angle between the curve and the x-axis at time $t = 0$.

10. (10) Identify and sketch the surface $x^2 + y^2 = z^2 - 1$.

11. (20) Find and classify the critical points of $f(x, y) = (y^2 - x^2)(y - 1) = y^3 - y^2 - x^2 y + x^2$.

12. (20) Find the absolute maximum and minimum of $f(x, y) = x^2 + 2y^2 - x - y$ on the square $0 \le x \le 1$, $0 \le y \le 1$.

13. Let V be the solid region which lies above the disk $x^2 + y^2 \leq 1$ in the xy-plane and below $z = \sqrt{x^2 + y^2 + 1} = \sqrt{r^2 + 1}$.

   (a) (10) Find the volume of V.

   (b) (10) Suppose $\vec{F} = z\vec{k} = (0, 0, z)$. Compute $\iint_{\mathcal{S}} \vec{F} \cdot \vec{n}\, dS$, where $\mathcal{S}$ is the boundary surface of V. (Hint: $\vec{F} \cdot \vec{n} = 0$ on the base $z = 0$ of V, and also on the vertical sides of V, so the only nonzero part of this integral occurs on the top surface $z = \sqrt{r^2 + 1}$.

14. (10) Let

$$f(x, y) = \begin{cases} \frac{x+2y}{x^2-y^2} & \text{if } x^2 \neq y^2 \\ 0 & \text{if } x^2 = y^2 \end{cases}$$

   Is f continuous at $(0, 0)$? Why or why not?

15. Let $z = x^2 - 3xy$.

   (a) (10) Find an equation for the tangent plane at $(x, y) = (1, 1)$.

   (b) (5) In what direction is z increasing most rapidly at $(x, y) = (1, 1)$.

   (c) (5) How fast is z increasing in the direction $(\frac{1}{2}, \frac{\sqrt{3}}{2})$ at $(x, y) = (1, 1)$.

   (d) (10) Find an equation for the tangent line to the level curve of $x^2 - 3xy$ which passes through $(x, y) = (1, 1)$.

   (e) (10) If the coordinate transformation $(x, y) = F(u, v)$ has

$$\frac{\partial x}{\partial u} = 1, \quad \frac{\partial x}{\partial v} = 2, \quad \frac{\partial y}{\partial u} = 3, \quad \frac{\partial y}{\partial v} = 4$$

   express $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$ as functions of x and y.

16. Let R be the region $0 \leq x \leq 2$, $0 \leq y \leq x^2$.

   (a) (5) Find the area of R.

   (b) (10) Find the centroid of R.

   (c) (5) Find the integral of $\sqrt{6y - 2y^{3/2}}$ over R.

——— The End ———