R. Bruner

Math 2030, Winter 2011, Test 3

April 4, 2011

1. Polar and cylindrical coordinates: Draw the diagram which shows the relation between (x, y) and (r, θ). Write the formulas for
   (a) x and y in terms of r and θ,
   (b) for $dA = dx\,dy$ in terms of r, θ, dr and $d\theta$,
   (c) for $dV = dx\,dy\,dz$ in terms of r, θ, z, dr, $d\theta$ and dz.

2. Spherical coordinates: Draw the diagram which shows the relation between (r, z) and (ρ, ϕ). Write the formulas for
   (a) r and z in terms of ρ and ϕ,
   (b) for $dA = dr\,dz$ in terms of ρ, ϕ, $d\rho$ and $d\phi$,
   (c) for $dV = dx\,dy\,dz$ in terms of ρ, θ, ϕ, $d\rho$, $d\theta$ and $d\phi$.

3. Suppose that $1 \leq f(x, y) \leq 1 + x + y$ for all (x, y) in the first quadrant. What can you say about $\iint_R f\,dA$, if R is the rectangle $[0, 1] \times [0, 2]$?

4. Find the area and centroid of the region above $y = x^2$ and below $y = x$.

5. Find the area and centroid of the region $0 \leq r \leq \sin\theta$ in the first quadrant.

6. Let $(x, y) = (u/v, uv)$.
   (a) Find the Jacobian $\dfrac{\partial(x, y)}{\partial(u, v)}$.
   (b) Find the Jacobian $\dfrac{\partial(u, v)}{\partial(x, y)}$ in terms of x and y.

7. Find the volume of the region described by $0 \leq r \leq z - z^2$ in cylindrical coordinates.

8. Let R be the same region as in the preceding problem. Find the moment of inertia of R about the z-axis.

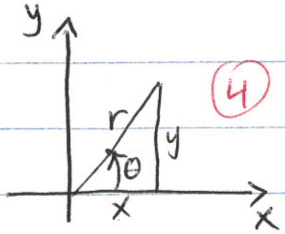
9. Find the volume of the region described by $\rho = \cos^2(\phi)$, $0 \leq \phi \leq \pi/2$, in spherical coordinates.

10. Find the z-coordinate, $\bar{z}$, of the centroid of the solid in the preceding problem.

——— The End ———

Test 3 Math 2030 W'11 Solutions

1.



(4)

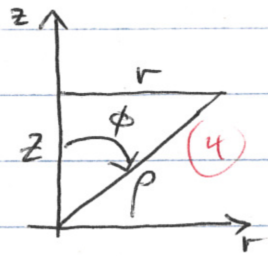
(a) $x = r\cos\theta$, $y = r\sin\theta$ (2)

(b) $dA = r\,dr\,d\theta$ (2)

(c) $dV = r\,dr\,d\theta\,dz$ (2)

10

2.



(4)

(a) $r = \rho\sin\phi$, $z = \rho\cos\phi$ (2)

(b) $dr\,dz = \rho\,d\rho\,d\phi$ (2)

(c) $dx\,dy\,dz = \rho^2\sin\phi\,d\rho\,d\theta\,d\phi$ (2)

10

3. $1 \le f(x,y) \le 1+x+y$ implies

10

$$\iint_R 1\,dA \le \iint_R f(x,y)\,dA \le \iint_R 1+x+y\,dA$$

$$\iint_R 1\,dA = \text{Area}(R) = 2$$

$$\iint_R 1+x+y\,dA = \int_0^1\int_0^2 1+x+y\,dy\,dx = \int_0^1 y+xy+\frac{y^2}{2}\Big|_0^2 dx$$

$$= \int_0^1 2+2x+2\,dx = x^2+4x\Big|_0^1 = 5$$

so

$$2 \le \iint_R f(x,y)\,dA \le 5.$$

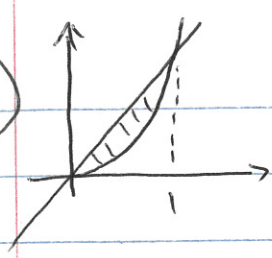
OR $$\int_0^2\int_0^1 1+x+y\,dx\,dy = \int_0^2 x+\frac{x^2}{2}+xy\Big|_{x=0}^{x=1} dy$$

$$= \int_0^2 \frac{3}{2}+y\,dy = \frac{3}{2}y+\frac{y^2}{2}\Big|_0^2$$

$$= 3+2 = 5$$

Points: Region = 2 Integrands = 2 Each Integral = 2

4.



(a) $\boxed{\text{Area}} = \int_0^1 x - x^2\,dx = \frac{x^2}{2} - \frac{x^3}{3}\Big|_0^1 = \frac{1}{2} - \frac{1}{3} = \boxed{\frac{1}{6}}$

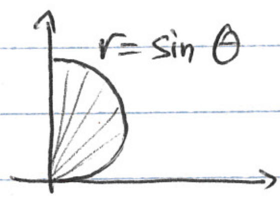
(b) $\iint x\,dA = \int_0^1 \int_{x^2}^{x} x\,dy\,dx = \int_0^1 xy\Big|_{y=x^2}^{y=x} dx$

$= \int_0^1 x^2 - x^3\,dx = \frac{x^3}{3} - \frac{x^4}{4}\Big|_0^1 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$ so $\boxed{\bar{x} = \frac{1/12}{1/6} = \frac{1}{2}}$

(c) $\iint y\,dA = \int_0^1 \int_{x^2}^{x} y\,dy\,dx = \int_0^1 \frac{1}{2}y^2\Big|_{y=x^2}^{y=x} dx = \int_0^1 \frac{x^2}{2} - \frac{x^4}{2}\,dx$

$= \frac{x^3}{6} - \frac{x^5}{10}\Big|_0^1 = \frac{1}{6} - \frac{1}{10} = \frac{4}{60} = \frac{1}{15}$ so $\boxed{\bar{y} = \frac{1/15}{1/6} = \frac{6}{15} = \frac{2}{5}}$

5.



(a) $\boxed{\text{Area}} = \int_0^{\pi/2} \int_0^{\sin\theta} r\,dr\,d\theta = \int_0^{\pi/2} \frac{r^2}{2}\Big|_{r=0}^{r=\sin\theta} d\theta$

$= \int_0^{\pi/2} \frac{1}{2}\sin^2\theta\,d\theta = \frac{\theta \pm \sin\theta\cos\theta}{4}\Big|_0^{\pi/2} = \boxed{\frac{\pi}{8}}$

(b) $\iint_R x\,dA = \int_0^{\pi/2} \int_0^{\sin\theta} r\cos\theta\, r\,dr\,d\theta = \int_0^{\pi/2} \frac{1}{3}r^3\cos\theta\Big|_{r=0}^{r=\sin\theta} d\theta$

$= \int_0^{\pi/2} \frac{1}{3}\sin^3\theta\cos\theta\,d\theta = \frac{1}{3}\int_0^1 u^3\,du = \frac{1}{12}$ so $\boxed{\bar{x} = \frac{1/12}{\pi/8} = \frac{2}{3\pi}}$

$u = \sin\theta$

$du = \cos\theta\,d\theta$

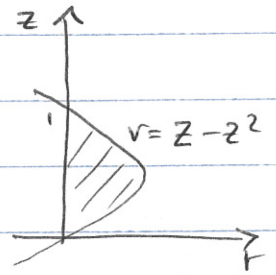
$\iint_R y\,dA = \int_0^{\pi/2} \int_0^{\sin\theta} r\sin\theta\, r\,dr\,d\theta = \int_0^{\pi/2} \frac{1}{3}r^3\sin\theta\Big|_{r=0}^{r=\sin\theta} d\theta$

$= \frac{1}{3}\int_0^{\pi/2} \sin^4\theta\,d\theta = \frac{1}{3}\left[\frac{3}{8}\theta + \frac{3}{8}\sin\theta\cos\theta - \frac{1}{4}\sin^3\theta\cos\theta\Big|_0^{\pi/2}\right]$

$= \frac{1}{3}\left[\frac{3}{8}\frac{\pi}{2}\right] = \frac{\pi}{16}$ so $\boxed{\bar{y} = \frac{\pi/16}{\pi/8} = \frac{1}{2}}$

6. (a) $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{1}{v} & \frac{-u}{v^2} \\ v & u \end{vmatrix} = 2\dfrac{u}{v}$

(b) $\dfrac{\partial(u,v)}{\partial(x,y)} = \dfrac{1}{\partial(x,y)/\partial(u,v)} = \dfrac{1}{2\left(\frac{u}{v}\right)} = \dfrac{1}{2x}$

7. 

$$Vol = \int_0^{2\pi}\int_0^1\int_0^{z-z^2} r\,dr\,dz\,d\theta$$

$$= 2\pi\int_0^1 \frac{1}{2}(z-z^2)^2\,dz = \pi\int_0^1 z^2-2z^3+z^4\,dz$$

$$= \pi\left[\frac{1}{3}-\frac{2}{4}+\frac{1}{5}\right] = \pi\left[-\frac{1}{6}+\frac{1}{5}\right] = \boxed{\frac{\pi}{30}}$$

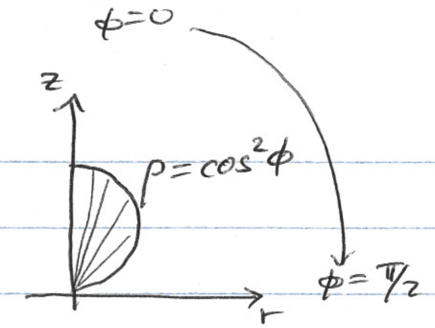
8. $$I_{z\text{-axis}} = \int_0^{2\pi}\int_0^1\int_0^{z-z^2} r^2\,r\,dr\,dz\,d\theta = 2\pi\int_0^1 \frac{1}{4}(z-z^2)^4\,dz$$

$$= \frac{\pi}{2}\int_0^1 z^4-4z^5+6z^6-4z^7+z^8\,dz$$

$$= \frac{\pi}{2}\left[\frac{1}{5}-\frac{4}{6}+\frac{6}{7}-\frac{4}{8}+\frac{1}{9}\right] = \frac{\pi}{2}\left[\frac{1}{5}-\frac{1}{2}+\frac{6}{7}-\frac{5}{9}\right]$$

$$= \frac{\pi}{2}\left[\frac{-3}{10}+\frac{54-35}{63}\right] = \frac{\pi}{2}\left[\frac{-3}{10}+\frac{19}{63}\right] = \frac{\pi}{2}\,\frac{-189+190}{630} = \boxed{\frac{\pi}{1260}}$$

9.



$$\text{Vol}=\int_0^{2\pi}\int_0^{\pi/2}\int_0^{\cos^2\phi}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

$$=2\pi\int_0^{\pi/2}\tfrac{1}{3}\sin\phi\cos^6\phi\,d\phi=\frac{2\pi}{3}\int_0^1 u^6\,du=\boxed{\frac{2\pi}{21}}$$

$u=\cos\phi$ $\quad u(0)=1$

$du=-\sin\phi\,d\phi$ $\quad u(\tfrac{\pi}{2})=0$

use minus sign to reverse limits

10.

$$\iiint z\,dV=\int_0^{2\pi}\int_0^{\pi/2}\int_0^{\cos^2\phi}\rho\cos\phi\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

$$=2\pi\int_0^{\pi/2}\frac{1}{4}\cos^8\phi\cos\phi\sin\phi\,d\phi=\frac{\pi}{2}\int_0^{\pi/2}\cos^9\phi\sin\phi\,d\phi$$

$$=\frac{\pi}{2}\int_0^1 u^9\,du$$ (same substitution as above)

$$=\frac{\pi}{20}$$ so $\boxed{\bar{z}=\frac{\pi/20}{2\pi/21}=\frac{21}{2}\,\frac{1}{20}=\frac{21}{40}}$