Test 2 W'11 M2030 Solutions

1. $f(x,y) = x^2\cos(y)$

$f_x = 2x\cos(y)$ $\qquad f_{xx} = 2\cos(y)$

$f_y = -x^2\sin(y)$ $\qquad f_{xy} = -2x\sin(y)$

10 $\qquad f_{yy} = -x^2\cos(y)$

2. $f(x,y) = \dfrac{xy}{x+y}$

(a) $z = f(1,2) = \dfrac{2}{3}$

$f_x = \dfrac{y(x+y) - xy(1)}{(x+y)^2} = \dfrac{y^2}{(x+y)^2}$ $\qquad f_x(1,2) = \dfrac{4}{9}$

$f_y = \dfrac{x^2}{(x+y)^2}$ by symmetry $\qquad f_y(1,2) = \dfrac{1}{9}$

Tangent plane at $(1,2,\tfrac{2}{3})$ is then

10 $$z - \frac{2}{3} = \frac{4}{9}(x-1) + \frac{1}{9}(y-2)$$

(b) f increases most rapidly in the direction of

5 $\nabla f(1,2) = (\tfrac{4}{9}, \tfrac{1}{9})$ (or $(4,1)$ is just as good)

5 (c) $D_{\vec{u}} f = 0$ if $\vec{u} \perp \nabla f$ so $\vec{u} = (-1,4)$ or $(1,-4)$ give directions in which f neither increases nor decreases.

3. (a) $\nabla(xy^2+yz^3) =$

$(y^2, 2xy+z^3, 3yz^2)$ which is $(4,5,6)$ at $(x,y,z)=(1,2,1)$.

The tangent plane to $xy^2+yz^3=6$ at $(1,2,1)$ is then

10 $$4(x-1)+5(y-2)+6(z-1)=0$$

(b) The curve is tangent to both tangent planes, hence it is perpendicular to both normal vectors.

5 The cross-product accomplishes this:

$$\begin{vmatrix} i & j & k \\ 4 & 5 & 6 \\ 1 & -1 & 1 \end{vmatrix} = (11, -(-2), -9) = (11, 2, -9)$$

4. $\dfrac{\partial f}{\partial x} = y^2$ and $\dfrac{\partial f}{\partial y} = 2xy$ so

$$\frac{\partial f}{\partial s} = y^2\cdot\frac{\partial x}{\partial s} + 2xy\frac{\partial y}{\partial s} \quad \text{and} \quad \frac{\partial f}{\partial t} = y^2\frac{\partial x}{\partial t} + 2xy\frac{\partial y}{\partial t}$$

10 $$= y^2 + 6xy \qquad\qquad = 2y^2 + 8xy$$

5 5. (a) $dC = y^2z^3\,dx + 2xyz^3\,dy + 3xy^2z^2\,dz$

5 (b) $\dfrac{dC}{C} = \dfrac{dx}{x} + 2\dfrac{dy}{dy} + 3\dfrac{dz}{z}$ so $\left|\dfrac{dC}{C}\right| \le .01 + 2(.01) + 3(.01)$

$= .06$

6. $xyz + xy + yz = 3$. Therefore, applying $\frac{\partial}{\partial x}$ gives

$$(yz+y) + (xy+y)\frac{\partial z}{\partial x} = 0 \quad \text{so} \quad \boxed{\frac{\partial z}{\partial x} = -\frac{y(z+1)}{y(x+1)}} = -\frac{z+1}{x+1}$$

and applying $\frac{\partial}{\partial y}$ gives

5

$$(xz+x+z) + (xy+y)\frac{\partial z}{\partial y} = 0, \quad \text{so} \quad \boxed{\frac{\partial z}{\partial y} = -\frac{xz+x+z}{xy+y}}$$

7. Find and classify the critical points of $4y - y^2 + x^2y - 4x^2 = f(x,y)$

$f_x = 2xy - 8x = 2x(y-4)$ so $x=0$ or $y=4$ at a C.P.

$f_y = 4 - 2y + x^2$

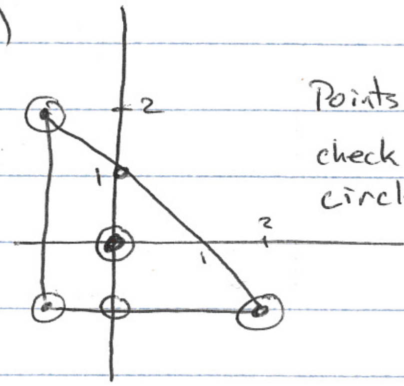
If $x=0$ then $f_y = 4-2y = 0$ gives $y=2$

If $y=4$ then $f_y = -4+x^2 = 0$ gives $x = \pm 2$.

10

	$(0,2)$	$(2,4)$	$(-2,4)$
$f_{xx} = 2y-8$	-4	0	0
$f_{xy} = 2x$	0	4	-4
$f_{yy} = -2$	-2	-2	-2
D	8	-16	-16
Type	max	saddle	saddle

8. Find abs max/min of x^2y+y^3 on the triangle with vertices $(-1,-1)$, $(-1,2)$ and $(2,-1)$



Points to check are circled.

10

Interior C.P.

$f_x = 2xy$ so $x=0$ or $y=0$

$f_y = x^2+3y^2$ so $x=y=0$

The only interior C.P. is $(0,0)$

Edge $x=-1$: $f = y+y^3$ so $f_y = 1+3y^2 > 0$ so f is increasing

Min at $(-1,-1)$, Max at $(-1,2)$

Edge $y=-1$: $f = -x^2-1$ so $f_x = -2x$. There is a C.P. at $x=0$. $(0,-1)$

Edge $x+y=1$: $f = (1-y)^2y + y^3 = y - 2y^2 + 2y^3$ so

$f_y = 1 - 4y + 6y^2$ Disc. $b^2-4ac = 16-24 < 0$ so no C.P.

Corners $f(-1,-1) = -1-1 = -2$

$f(-1,2) = 2+8 = 10$ Maximum 10 at $(-1,2)$

$f(2,-1) = -4-1 = -5$ Minimum -5 at $(2,-1)$

$f(0,0) = 0$

$f(0,-1) = -1$

9. $L = xy + 2xz + 3yz - \lambda(xyz - 6)$

At a critical point,

$L_x = y + 2z - \lambda(yz)$ $\qquad \lambda xyz = xy + 2xz$

$L_y = x + 3z - \lambda(xz)$ $\qquad = xy + 3yz$

4 $L_z = 2x + 3y - \lambda(xy)$ $\qquad = 2xz + 3yz$ 7

Combining these too at a time gives $\quad 2x = 3y$

$x = 3z$

$y = 2z$

10

So $xyz = (3z)(2z)(z) = 6z^3 = 6$, so $z = 1,\ y = 2,\ x = 3.$

---

w/o Lagrange Multipliers: $z = \frac{6}{xy}$ so

$$C = xy + \frac{12x}{xy} + \frac{18y}{xy} = xy + \frac{12}{y} + \frac{18}{x}$$

Then $C_x = y - \frac{18}{x^2}$ $\quad$ so $\quad y = 18/x^2$

$C_y = x - \frac{12}{y^2}$ $\qquad x = 12/y^2$

(Not!) Simplest: $x^2y = 18$ so $x^2y^2 = 18y$

$xy^2 = 12$ $\qquad x^2y^2 = 12x$ $\quad$ so $18y = 12x$

so $y = \frac{2}{3}x$ gives $\frac{2}{3}x^3 = 18$ or $x^3 = \frac{3 \cdot 18}{2} = 3 \cdot 9 = 27$. Hence $x = 3,\ y = 2$ and $z = 6/6 = 1.$

Best ~~Slower~~: $y = \frac{18}{x^2} = \frac{18}{(12/y^2)^2} = \frac{18y^4}{12^2}$ so $12^2 y - 18y^4 = 0$

or $8y - y^4 = 0$. y cannot be 0 so $y^3 = 8$, $y = 2$. Then $x = \frac{12}{4}$ $= 3$ while $z = 6/6 = 1$.