Math 2030 Winter 2011

Test 1 Solutions

1. Find the intersection of the line through $(1,2,1)$ and $(2,1,2)$ with the plane $2x-z=3$.

(5) The line has direction $(2,1,2)-(1,2,1)=(1,-1,1)$ so can be written

(1) $$\ell(t)=(1,2,1)+t(1,-1,1)=(1+t,2-t,1+t)$$

Intersecting with the plane requires that

$$2x-z=2(1+t)-(1+t)=3$$

or $1+t=3$ or $t=2$.

(4) This is the point $\boxed{\ell(2)=(3,0,3).}$

2. The plane through $(5,1,4)$, $(1,-2,3)$ and $(2,1,0)$ has normal perpendicular to all vectors in the plane, hence parallel to the cross-product of any two such:

$$(5,1,4)-(2,1,0)=(3,0,+4)$$

(4) $$(2,1,0)-(1,-2,3)=(1,3,-3)$$ $(5,1,4)-(1,-2,3)=(4,3,1)$

so our normal can be given as

(3) $$\begin{vmatrix} i & j & k \\ 3 & 0 & +4 \\ 1 & 3 & -3 \end{vmatrix} = (-12,-(-13),9)=(-12,13,9)$$

Evaluating at any one of the points gives -11, so the plane is

(3) $$\boxed{-12x+13y+9z=-11.}$$

OR

$$\boxed{12x-13y-9z=11}$$

3. (a) $x^2-2y^2-3z^2=4$ is $x^2-4=2y^2+3z^2$

Cross-sections ellipse, hyperbola, hyperbola, nonexistent when

(5) $x^2<4 \Rightarrow$ $\boxed{\text{hyperboloid of two sheets.}}$

(b) $x^2+2y^2+3z^2=4$ all positive coefficients

(5) $\Rightarrow$ $\boxed{\text{ellipsoid}}$

4. Find the projection of $(11,1,7)$ onto $(2,1,1)$ and decompose $(11,1,7)$ as the sum of a vector parallel to $(2,1,1)$ and a vector perpendicular to $(2,1,1)$:

$$\text{proj}_{(2,1,1)}(11,1,7) = \frac{(11,1,7)\cdot(2,1,1)}{(2,1,1)\cdot(2,1,1)}(2,1,1)$$

(5)
$$= \frac{22+1+7}{4+1+1}(2,1,1) = \frac{30}{6}(2,1,1) = 5(2,1,1) = \boxed{(10,5,5)}$$

(15)
$$\boxed{(11,1,7) = \underbrace{\underbrace{(10,5,5)}_{\text{parallel}} + \underbrace{(1,-4,2)}_{\text{perpendicular}}}_{\text{to }(2,1,1)}}$$

since it is $5\cdot(2,1,1)$ and $(1,-4,2)\cdot(2,1,1) = 2-4+2=0$.

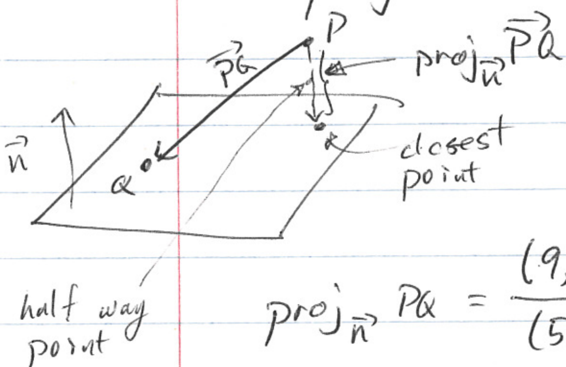
5. The line of intersection of $2x-y+z=1$ and $x+y+z=1$ has direction perpendicular to both planes' normals:

(7)
$$\begin{vmatrix} i & j & k \\ 2 & -1 & 1 \\ 1 & 1 & 1 \end{vmatrix} = (-2, -(1), 3).$$

The point $(0,0,1)$ is in both,

(13) so the line is $l(t) = (0,0,1)+t(-2,-1,3) = \boxed{(-2t, -t, 1+3t).}$

(6.) (a) Find the point of the plane $5x - y - z = 2$ closest to $P = (-9, 4, 3)$ by taking any point Q in the plane and projecting $\overrightarrow{PQ}$ onto the normal $(5, -1, -1)$, & ~~subtracting~~ adding this projection ~~from~~ to $(-9, 4, 3)$:



Let $Q = (0, 0, -2)$. Then $PQ = (9, -4, -5)$

$$\text{proj}_{\vec{n}} PQ = \frac{(9,-4,-5) \cdot (5,-1,-1)}{(5,-1,-1) \cdot (5,-1,-1)} (5, -1, -1)$$

$\vec{PQ}$ (5)

proj (5)

$$= \frac{45 + 4 + 5}{25 + 1 + 1} (5, -1, -1) = 2(5, -1, -1) = (10, -2, -2)$$

The closest point is therefore

$$(-9, 4, 3) + (10, -2, -2) = \boxed{(1, 2, 1).}$$

(5) (b) To get $\frac{1}{2}$ way to the plane, add half of this:

$$(-9, 4, 3) + (5, -1, -1) = \boxed{(-4, 3, 2).}$$

<u>Checks</u>: $(5,-1,-1) \cdot (1,2,1) = 5 - 2 - 1 = 2$ so $(1,2,1)$ is in the plane. Also, the average of $(-9, 4, 3)$ and $(1, 2, 1)$ is $(-4, 3, 2)$.

(7.) The line through $(1,3,3)$ and $(2,3,6)$ is $\ell(t) =$

(5) $(1,3,3) + t(2-1, 3-3, 6-3) = (1+t, 3, 3+3t)$. Then $z = 3 + 3t$ while $xy = (1+t) \cdot 3 = 3 + 3t$ also, so this line lies in

(5) $z = xy$.

<u>NOTE</u>: The lines $\ell_1(t) = (a, bt, abt)$ and $\ell_2(t) = (at, b, abt)$ lie in $z = xy$ and pass through (a, b, ab), so this surface is made entirely of straight lines!

(8) $\vec{r}(t) = (t^2-t,\ t^3+t,\ t^2-t^3)$

(5) (a) $\vec{v}(t) = \boxed{(2t-1,\ 3t^2+1,\ 2t-3t^2)}$

(5) (b) $\vec{a}(t) = \boxed{(2,\ 6t,\ 2-6t)}$

(c) $\vec{a}(1) = (2,6,-4)$ and the tangent $\vec{v}(1) = (1,4,-1)$

so the tangential component of $\vec{a}(1)$ is

$$\vec{a}_T = \frac{(2,6,-4)\cdot(1,4,-1)}{(1,4,-1)\cdot(1,4,-1)}(1,4,-1) = \frac{2+24+4}{1+16+1}(1,4,-1)$$

(3) $$= \frac{30}{18}(1,4,-1) = \boxed{\frac{5}{3}(1,4,-1)}$$

The normal component is then $\vec{a}_N = \vec{a}(1) - \vec{a}_T = ($

(2) $$= (2,6,-4) - \frac{5}{3}(1,4,-1) = \frac{1}{3}(6-5,\ 18-20,\ -12+5) = \boxed{\frac{1}{3}(1,-2,-7)}$$

(5) (d) $\frac{ds}{dt}(1) = |\vec{v}(1)| = \sqrt{1+16+1} = \boxed{\sqrt{18}}$

(e) $|\vec{a}_N| = v^2 K$ so $K = |\vec{a}_N| / \sqrt{18}^2 = \frac{1}{3\cdot 18}\sqrt{1+4+49}$

(5) $$= \frac{\sqrt{54}}{54} = \boxed{\frac{1}{\sqrt{54}}} = \frac{1}{3\sqrt{6}}$$

(f) The tangent line at $t=1$ passes through $\vec{r}(1)$ in direction $\vec{v}(1)$, so has parameterization

(2)

(2) $$\ell(t) = (0,2,0) + t(1,4,-1) = \boxed{(t,\ 2+4t,\ -t).}$$

(1)