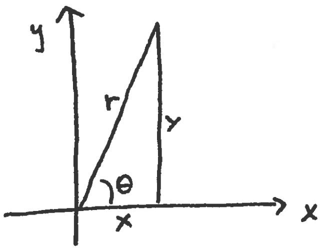
M 2030 F17 Test 3 Solutions

1.

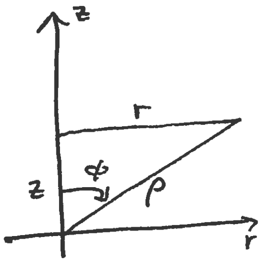


(a) $x = r\cos\theta$
$y = r\sin\theta$

(b) $dx\,dy = r\,dr\,d\theta$

(c) $dx\,dy\,dz = r\,dr\,d\theta\,dz$

2.

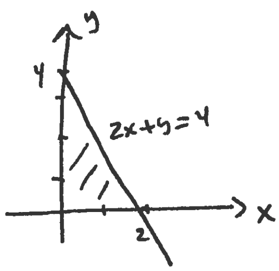


(a) $r = \rho\sin\phi$
$z = \rho\cos\phi$

(b) $dr\,dz = \rho\,d\rho\,d\phi$

(c) $dx\,dy\,dz = \rho^2\sin\phi\,d\rho\,d\phi\,d\theta$

3.



Base

$0 \le x \le 2$

$0 \le y \le 4-2x$

$0 \le z \le 2x+4y$

$$\text{Vol} = \int_0^2 \int_0^{4-2x} \int_0^{2x+4y} dz\,dy\,dx$$

$$= \int_0^2 \int_0^{4-2x} 2x+4y\,dy\,dx = \int_0^2 2xy + 2y^2 \Big|_0^{4-2x} dx$$

$$= \int_0^2 2x(4-2x) + 2(4-2x)^2\,dx$$

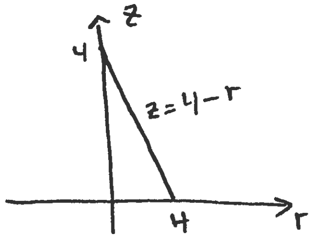
$$= \int_0^2 8x - 4x^2 + 32 - 32x + 8x^2\,dx$$

$$= \int_0^2 32 - 24x + 4x^2\,dx = \left[32x - 12x^2 + \tfrac{4}{3}x^3 \Big|_0^2\right]$$

$$= 64 - 48 + \frac{32}{3} - 0 = 16 + \frac{32}{3} = 16\left(1 + \frac{2}{3}\right) = \boxed{\frac{80}{3}}$$

4.

(9:37)



(Not to scale)

$0 \le \theta \le 2\pi$

$0 \le r \le 4$

$0 \le z \le 4-r$

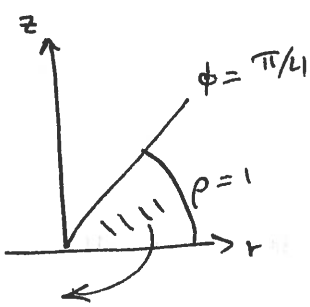
$$\text{Vol} = \int_0^{2\pi}\int_0^4\int_0^{4-r} r\,dz\,dr\,d\theta = 2\pi\int_0^4 rz\Big|_{z=0}^{z=4-r} dr$$

$$= 2\pi\int_0^4 4r - r^2\,dr = 2\pi\left[2r^2 - \frac{r^3}{3}\Big|_0^4\right]$$

$$= 2\pi\left[32 - \frac{64}{3}\right] = 64\pi\left[1 - \frac{2}{3}\right] = \boxed{\frac{64\pi}{3}}$$

5.

(9:40)



$\frac{\pi}{4} \le \phi \le \frac{\pi}{2}$

$0 \le \theta \le 2\pi$

$0 \le \rho \le 1$

$$\text{Vol} = \int_0^{2\pi}\int_{\pi/4}^{\pi/2}\int_0^1 \rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

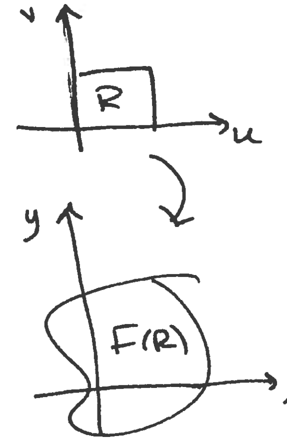
$$= \left(\int_0^{2\pi} d\theta\right)\left(\int_{\pi/4}^{\pi/2}\sin\phi\,d\phi\right)\left(\int_0^1 \rho^2\,d\rho\right)$$

$$= 2\pi\left(-\cos\phi\Big|_{\pi/4}^{\pi/2}\right)\left(\frac{1}{3}\right)$$

$$= \frac{2\pi}{3}\left(-0 + \frac{1}{\sqrt{2}}\right) = \boxed{\frac{\pi\sqrt{2}}{3}}$$

⑥ (9:43)

$x = u + v^2$
$y = v - u^2$



see below (#9) for a "better" drawing.

(a) $$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1 & 2v \\ -2u & 1 \end{vmatrix} = \boxed{1+4uv}$$

(b) $$\text{Area}\,(F(R)) = \iint_{F(R)} dx\,dy$$

$$= \int_0^1 \int_0^1 (1+4uv)\,du\,dv$$

$$= \int_0^1 u + 2u^2 v \Big|_{u=0}^{u=1} dv = \int_0^1 1 + 2v - 0 \; dv$$

$$= v + v^2 \Big|_0^1 = 1 + 1 - 0 = \boxed{2}$$

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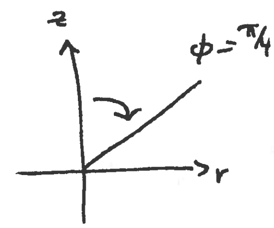
⑦ (9:45)

(a) $\rho = 2$
$r^2 + z^2 = 4$
$x^2 + y^2 + z^2 = 4$

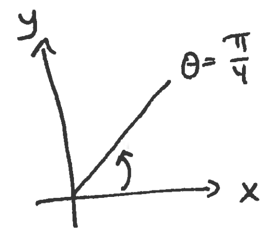
(b) $\phi = \pi/4$
$r = z$
$x^2 + y^2 = z^2$



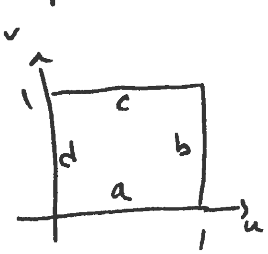
⑧ (9:46)

(a) $r = 3$
$x^2 + y^2 = 9$

(b) $\theta = \pi/4$
$y = x$



9.



a: $(0,0) \mapsto (0,0)$
$(1,0) \mapsto (1,-1)$
$v = 0$
$0 \le u \le 1$
$x = u$
$y = -u^2$

b: $(1,1) \mapsto (2,0)$
$u = 1$
$0 \le v \le 1$
$x = 1 + v^2$
$y = v - 1$
so $v = y+1$ and
$x = 1 + (y+1)^2$

c: $(0,1) \mapsto (1,1)$
$v = 1$
$0 \le u \le 1$
$x = u + 1$
$y = 1 - u^2$
$u = x - 1$
$y = 1 - (x-1)^2$

d: $u = 0$
$0 \le v \le 1$
$x = +v^2$
$y = v$