Test 1 M 2030 F17 Solutions

(1.) The line through $(1,3,3)$ and $(0,4,4)$ is $\vec{\ell}(t) = t(1,3,3) + (1-t)(0,4,4)$

$= (t,\ 3t+4-4t,\ 3t+4-4t) = (t, 4-t, 4-t)$. This intersects the plane

5 $3x - 2y - z = 12$ when $3t - 2(4-t) - (4-t) = 12$, or

$3t - 3(4-t) = 12$, i.e.

$t - 4 + t = 4$

$2t = 8$

$t = 4$

5 The intersection point is $\boxed{\vec{\ell}(4) = (4,0,0).}$

(2.) Two vectors parallel to the plane are $\overrightarrow{(1,2,3)}$, the direction of the line, and $(2,2,0) - (1,1,1) = (1,1,-1)$, a vector from the line to $(2,2,0)$. Their cross-product will be normal to the plane:

$$\vec{n} = \begin{vmatrix} i & j & k \\ 1 & 2 & 3 \\ 1 & 1 & -1 \end{vmatrix} = (-5, -(-4), -1) = (-5, 4, -1)$$

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The plane is then $5x - 4y + z = c$ for some c. At $(1,1,1)$ we get $c = 2$, so 5

$$\boxed{5x - 4y + z = 2}$$

(3.) The cross product of the normals will be parallel to the line

$$\vec{v} = \begin{vmatrix} i & j & k \\ 1 & 1 & 0 \\ 0 & 1 & 2 \end{vmatrix} = (2, -(2), 1)$$

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The point $(1,1,1)$ lies on both planes: $\boxed{\ell(t) = (1,1,1) + t(2,-2,1)}$

5 OR $(0, 2, \tfrac{1}{2})$ (@ $x=0$)

OR $(-1, 3, 0)$ (@ $z=0$)

OR $(2, 0, \tfrac{3}{2})$ (@ $y=0$).

4. The line through $(5,-3,9)$ normal to the plane is

$$\vec{\ell}(t) = (5,-3,9) + t(1,-1,2) = (5+t, -3-t, 9+2t).$$

(a) It intersects $x - y + 2z = 2$ if

$$5+t-(-3-t)+2(9+2t) = 2$$

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i.e. $5+3+18+t+t+4t = 2$

i.e. $24+6t = 0$, so $t=-4$ and $\boxed{\vec{\ell}(-4) = (1,1,1).}$

(b) Distance to the plane is the length of the normal vector needed to get there: $|-4(1,-1,2)| = 4\sqrt{1+1+4} = \boxed{4\sqrt{6}}.$

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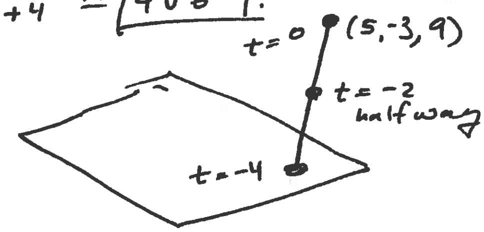

(c) Half way there we are at

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$$\boxed{\ell(-2) = (3,-1,5)}$$

This is also the midpoint of the segment from $(5,-3,9)$ to $(1,1,1)$, i.e. $\left(\frac{5+1}{2}, \frac{-3+1}{2}, \frac{9+1}{2}\right)$.

5. $\vec{r}(t) = (t, 1/t, t^2)$

(a) $\boxed{\vec{v}(t) = (1, -1/t^2, 2t)}$

(b) $\boxed{\vec{a}(t) = (0, 2/t^3, 2)}$

(c) $\vec{a}(1) = (0,2,2)$ and the tangential direction is $\vec{v}(1) = (1,-1,2)$

Hence $\vec{a}_T = \text{proj}_{(1,-1,2)}(0,2,2) = \frac{0-2+4}{1+1+4}(1,-1,2) = \frac{1}{3}(1,-1,2) = \vec{a}_T$

$\vec{a}_N = \vec{a} - \vec{a}_T = \left(0-\frac{1}{3}, 2-(-\frac{1}{3}), 2-\frac{2}{3}\right) = \left(-\frac{1}{3}, \frac{7}{3}, \frac{4}{3}\right)$

5 ea

$$\boxed{\vec{a} = \underbrace{\tfrac{1}{3}(1,-1,2)}_{\vec{a}_T} + \underbrace{\tfrac{1}{3}(-1,7,4)}_{\vec{a}_N}}$$

(d) $\frac{ds}{dt} = |\vec{v}(1)| = \sqrt{1+1+4} = \boxed{\sqrt{6}}$

(e) $v^2 K = |\vec{a}_N| = \frac{1}{3}\sqrt{1+49+16} = \frac{1}{3}\sqrt{66}$ so $\boxed{K = \frac{\sqrt{66}/3}{6} = \frac{\sqrt{66}}{18}}$

(f) $\vec{r}(1) = (1,1,1)$

$\vec{v}(1) = (1,-1,2)$ so tangent line is $\boxed{\vec{\ell}(t) = (1,1,1) + t(1,-1,2)}$

6.

$$\operatorname{proj}_{(1,1,1)}(1,3,5) = \frac{1+3+5}{1+1+1}(1,1,1) = 3(1,1,1)$$

$$(1,3,5) = \underbrace{3(1,1,1)}_{\parallel \text{ to } (1,1,1)} + \underbrace{(-2,0,2)}_{\perp \text{ to } (1,1,1)}$$

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7. The line through $(3,1,5)$ in the direction $\overrightarrow{(2,1,4)}$ can be written

5 $$\vec{r}(t) = \overrightarrow{(3,1,5)} + t\,\overrightarrow{(2,1,4)} = (3+2t,\ 1+t,\ 5+4t)$$

On the line,

$$\begin{aligned} x+2y &= 3+2t+2(1+t) \\ &= 5+4t \\ &= z \end{aligned}$$

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so it lies in this plane.

Alternative proof: At $(3,1,5)$, $x+2y = 3+2 = 5 = z$, so $(3,1,5)$ lies in the plane. The direction $(2,1,4)$ satisfies $(2,1,4)\cdot(1,2,-1) = 2+2-4 = 0$, so is perpendicular to $\vec{n} = (1,2,-1)$, the normal to the plane. Hence, the entire line lies in the plane.

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