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Math 2030, Winter 2017, Quiz 14
11 December 2017
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No calculators needed or allowed.

Let $\mathcal{V}$ be the cylindrical solid given by $0 \le z \le 4$ and $0 \le r \le 2$ in cylindrical coordinates. Observe that the boundary surface of $\mathcal{V}$ has three parts: a top disk $\mathcal{T}_1$, a bottom disk $\mathcal{T}_0$, and the lateral side surface $\mathcal{S}$ given by $r = 2$, $0 \le z \le 4$.

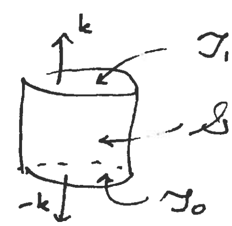
Let $\mathbf{F}(x, y, z) = (x, y, 0)$.

1. Show that $\mathbf{F} \cdot \mathbf{n} = 0$ on $\mathcal{T}_0$ and $\mathcal{T}_1$, so that

$$\iint_{\partial\mathcal{V}} \mathbf{F} \cdot \mathbf{n}\, \mathbf{dS} = \iint_{\mathcal{S}} \mathbf{F} \cdot \mathbf{n}\, \mathbf{dS}$$

2. On a surface of revolution $r = f(z)$, we have $\mathbf{n}\ dS = (x, y, -f(z)f'(z))\, dz d\theta$. Use this to compute $\mathbf{n}\ dS$ for the lateral surface $\mathcal{S}$.

3. Compute $\iint_{\mathcal{S}} \mathbf{F} \cdot \mathbf{n}\ dS$.



4. Compute $\nabla \cdot \mathbf{F}$.

5. Compute $\iiint_{\mathcal{V}} \nabla \cdot \mathbf{F}\ dV$.

1. On $\mathcal{T}_1$ and $\mathcal{T}_0$, $\vec{n} = \vec{k}$ and $-\vec{k}$ respectively. Thus,
$\vec{F} \cdot \vec{n} = (x, y, 0) \cdot (0, 0, 1) = 0$.

2. $\vec{n}\, dS = (x, y, 0)\, dz\, d\theta$ since $r = f(z) = 2$ on $\mathcal{S}$, so $f'(z) = 0$.

3. $$\iint_{\mathcal{S}} \vec{F} \cdot \vec{n}\ dS = \int_0^4 \int_0^{2\pi} (x, y, 0) \cdot (x, y, 0)\, dz\, d\theta = \int_0^4\int_0^{2\pi} x^2 + y^2\ dz\, d\theta = \int_0^4\int_0^{2\pi} 4\ dz\, d\theta$$
$$= 4 \cdot 2\pi \cdot 4 = 32\pi$$

4. $\nabla \cdot F = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(0) = 1 + 1 = 2$

5. $$\iiint_{\mathcal{V}} \nabla \cdot F\ dV = \int_0^{2\pi}\int_0^4\int_0^2 2\ r\,dr\, dz\, d\theta = 2\pi \int_0^4 r^2\Big|_0^2\ dz = 2\pi \cdot 4 \cdot 4 = 32\pi$$

Comments:

The lateral surface is the surface of revolution of the constant function $r = f(z) = 2$. Thus, (z, θ) are the proper coordinates to use to describe it. Nonetheless, it is clearer to write $\vec{F} = (x, y, 0)$ and $\vec{n}\,dS = (x, y, 0)\,dz\,d\theta$,

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$$x = 2\cos\theta$$
$$y = 2\sin\theta$$
$$z = z$$

are the descriptions of x, y and z in terms of the parameters z and θ.

Similarly, inside V we have

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = z$$

to express x, y and z in terms of the cylindrial coordinates r, θ and z.

The relation $x^2 + y^2 = r^2$ (in V) and hence $x^2 + y^2 = 4$ (on S) gets you back to z and θ.