Math 2030 Fall 2017 Final

1. $f = 4x^3 - 2x^2y + y^2 + 8y$

C.P.:

$f_x = 12x^2 - 4xy = 4x(3x - y)$

$f_y = -2x^2 + 2y + 8 = 2(y - x^2 + 4)$

At a C.P.: $x = 0$ or $y = 3x$; $y = x^2 - 4$

If $x = 0$ then $y = -4$.

If $y = 3x$ then $3x = x^2 - 4$ or $x^2 - 3x - 4 = 0$

so $(x-4)(x+1) = 0$ and $x = 4$ or $x = -1$.

C.P.	$(0, -4)$	$(4, 12)$	$(-1, -3)$
$f_{xx} = 24x - 4y$	16	$4 \cdot 24 - 4 \cdot 12 = 48$	$-24 + 12 = -12$
$f_{xy} = -4x$	0	-16	4
$f_{yy} = 2$	2	2	2
$D = f_{xx}f_{yy} - f_{xy}^2$	> 0, $f_{xx} > 0$, min	$96 - 256 < 0$, saddle	< 0, saddle

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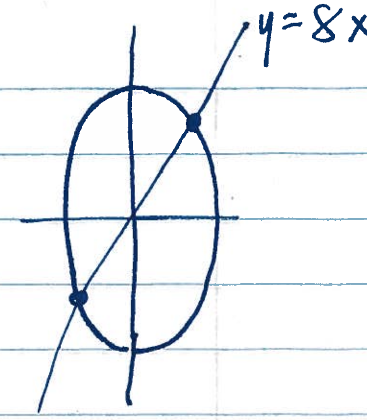

2. On the ellipse $4x^2+y^2=4$

(a) Find the max/min of $x+2y$.

$$L = x+2y - \lambda(4x^2+y^2-4)$$
$$L_x = 1 - \lambda(8x)$$
$$L_y = 2 - \lambda(2y)$$

C.P. $\Rightarrow \lambda = \frac{1}{8x} = \frac{2}{2y} = \frac{1}{y} \Rightarrow y = 8x$

and

$$4x^2+(8x)^2=4 \Rightarrow 4x^2+64x^2=4 \Rightarrow 17x^2=1$$

$\Rightarrow (x,y) = \pm(1/\sqrt{17}, 8/\sqrt{17})$

$$\boxed{\text{Max} = 17/\sqrt{17} = \sqrt{17}, \quad \text{min} = -\sqrt{17}}$$

(b) Tangent line at max: $\nabla(4x^2+y^2) = (8x, 2y)$

$= (8/\sqrt{17}, 16/\sqrt{17}) = \left(8/\sqrt{17}\right)\cdot(1,2)$ so tangent line

is $\boxed{\left(x-\frac{1}{\sqrt{17}}\right) + 2\left(y-\frac{8}{\sqrt{17}}\right) = 0}$

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$$(x,y) = (1-t^2,\ t^3-t) \qquad -1 \le t \le 1.$$

3. (a) $$\int_C x\,dy = \int_{-1}^{1} (1-t^2)(3t^2-1)\,dt$$

$$= \int_{-1}^{1} 3t^2-1-3t^4+t^2\,dt = \int_{-1}^{1} -3t^4+4t^2-1\,dt$$

$$= 2\int_{0}^{1} -3t^4+4t^2-1\,dt = 2\left[\frac{-3}{5}+\frac{4}{3}-1\right] = \frac{2}{15}(-9+20-15)$$

$$= \frac{2}{15}(-4) = \boxed{-8/15}$$

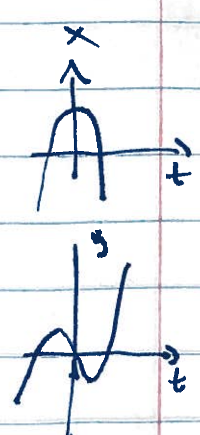
(b) $$\int_C y\,dx = \int_{-1}^{1} (t^3-t)(-2t)\,dt = \int_{-1}^{1} -2t^4+2t^2\,dt$$

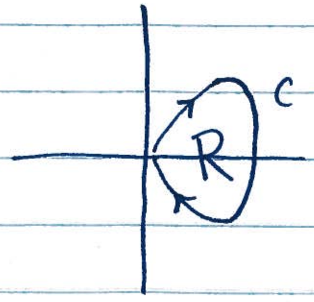
$$= 2\int_{0}^{1} -2t^4+2t^2\,dt = 2\left[-\frac{2}{5}+\frac{2}{3}\right] = \frac{4}{15}[5-3] = \boxed{\frac{8}{15}}$$

(c) $$\int_C y\,dx = \pm\iint_R -1\,dA \quad \text{so} \quad \text{Area}(R) = 8/15$$

(we do not know if C is oriented correctly)

[It appears that C traverses the boundary with the region R on its right, not its left.]





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(4.) $\vec{r}(t) = (t^2+4t,\ t^3-8t)$ $\vec{r}(2) = (12, -8)$

(a) $\vec{v} = \vec{r}\,'(t) = (2t+4,\ 3t^2-8)$

(b) $v(2) = |(8,4)| = 4|(2,1)| = 4\sqrt{5}$

(c)

$$\vec{\ell}(t) = (12,-8) + t(8,4)$$

(d) $\vec{a} = \vec{v}\,' = (2, 6t)$

(e) $\vec{a}(2) = (2,12)$

Tangential: $\dfrac{\vec{a}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\vec{v} = \dfrac{(2,12)\cdot(8,4)}{(8,4)\cdot(8,4)}(8,4)$

$$= \frac{16+48}{64+16}(8,4) = \frac{64}{80}(8,4) = \boxed{\frac{4}{5}(8,4) = \vec{a}_T}$$

Normal: $(2,12) - \frac{4}{5}(8,4) = \frac{1}{5}\left[(10,60) - (32,16)\right]$

$$= \frac{1}{5}(-22,44) = \boxed{\frac{22}{5}(-1,2) = \vec{a}_N}$$ Check: $\perp$ to $\vec{v}$!

(f) $\boxed{\kappa = \dfrac{|a_N|}{v^2} = \dfrac{22}{5}\sqrt{5}\Big/16\cdot5} = \dfrac{22\sqrt{5}}{16\cdot25} = \boxed{\dfrac{11\sqrt{5}}{200}}$

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(5.) $f_x = e^x \cos(2y)$ $f_{xx} = e^x\cos(2y)$

$f_y = -2e^x \sin(2y)$ $f_{xy} = f_{yx} = -2e^x \sin(2y)$

$f_{yy} = -4e^x \cos(2y)$

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(6.) $z = x^2y + xy^3 = 1^2\cdot 2 + 1\cdot 2^3 = 2+8 = 10$ at $(x,y) = (1,2)$

$z_x = 2xy + y^3 = 2\cdot 1\cdot 2 + 2^3 = 12$

$z_y = x^2 + 3xy^2 = 1^2 + 3\cdot 1\cdot 2^2 = 13$

so $\boxed{z - 10 = 12(x-1) + 13(y-2)}$

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(7.) $f(x,y,z) = xyz + x^2 + y^2 + z^2$ at $(3,2,1)$

(a) Most rapid incr $= \nabla f = (yz+2x, xz+2y, xy+2z)$

$= (2+6, 3+4, 6+2) = \boxed{(8,7,8)}$ at $(3,2,1)$

(b) Tangent plane is $\boxed{(8,7,8)\cdot(x-3, y-2, z-1) = 0}$

(c) $f_x + f_z \frac{\partial z}{\partial x} = 0$ so $\boxed{\frac{\partial z}{\partial x} = -f_x/f_z = -(yz+2x)/(xy+2z)}$

and $\boxed{\frac{\partial z}{\partial y} = -(xz+2y)/(xy+2z).}$

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8. $\frac{(4,3,4)\cdot(1,2,2)}{(1,2,2)\cdot(1,2,2)}(1,2,2) = \frac{4+6+8}{1+4+4}(1,2,2) = \frac{18}{9}(1,2,2)$

$= (2,4,4) =$ component parallel to $(1,2,2)$

Normal $= (4,3,4) - (2,4,4) = (2,-1,0)$

$\boxed{\underbrace{(4,3,4) = (2,4,4)}_{\text{parallel}} + \underbrace{(2,-1,0)}_{\text{perpendicular}} \text{ to } (1,2,2)}$

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9. (a) $f = \int y\,dx + g(y,z) = xy + g(y,z)$ gives

$f_y = x + g_y \overset{\text{want}}{=} x$ so $g_y = 0$ so $g(y,z) = h(z)$, and

$f = xy + h(z)$. Then $f_z = h'(z) \overset{\text{want}}{=} z$ so $h(z) = \frac{1}{2}z^2$

$\boxed{f = xy + \frac{1}{2}z^2 \text{ is the potential, } \vec{F} \text{ is conserv.}}$

(b) If $f_x = y$ then $f_{xy} = 1$
But $f_y = z$ gives $f_{yx} = 0$

$\boxed{\text{Not conservative.}}$

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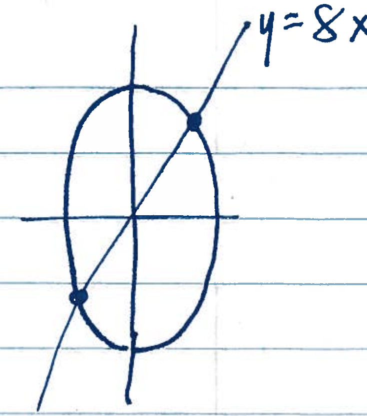

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(10.) $\int \nabla f \cdot d\vec{r} = f(\text{end}) - f(\text{begin}) = f(1,2) - f(2,3)$

$= 1^2 \cdot 2 - 2^2 \cdot 3 = 2 - 12 = \boxed{-10}$ $(f = x^2 y)$

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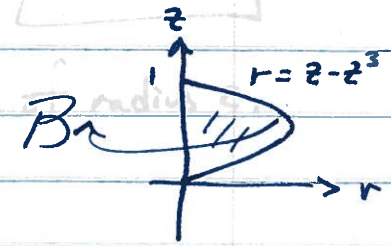

(11.) $\text{Vol}(B) = \int_0^{2\pi}\int_0^1\int_0^{z-z^3} r\,dr\,dz\,d\theta$

$= 2\pi \int_0^1 \frac{1}{2} r^2 \Big|_0^{z-z^3} dz = \pi \int_0^1 (z - z^3)^2\,dz$

$= \pi \int_0^1 z^2 - 2z^4 + z^6\,dz = \boxed{\pi\left[\frac{1}{3} - \frac{2}{5} + \frac{1}{7}\right]}$

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(12.) $\vec{F} = (x, y, 0)$ $\vec{n}\,dS = (x, y, 1-z)\,dz\,d\theta$

where $x = r\cos\theta, y = r\sin\theta$ and $r = 1 - z$

(a) So (a) $\vec{F} \cdot \vec{n}\,dS = x^2 + y^2 = r^2 = (1-z)^2\,dz\,d\theta$

$\iint_S \vec{F} \cdot \vec{n}\,dS = \int_0^{2\pi}\int_0^1 (1-z)^2\,dz\,d\theta = 2\pi \int_0^1 (z-1)^2\,dz = \boxed{\frac{2\pi}{3}}$

(b) $\boxed{\text{Bottom surface } z = 0 \text{ has } \vec{n} = -\vec{k} = (0,0,-1) \text{ so } \vec{F}\cdot\vec{n} = 0}$

(c) $\iiint_\gamma 1 + 1\,dV = 2\,\text{Vol}(\gamma) = \iint_S \vec{F}\cdot\vec{n}\,dS = \boxed{\frac{2\pi}{3} \text{ by (a)}}$

$\boxed{= 2\left(\frac{\pi}{3}\right) \text{ by Vol of a cone}}$

OR

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(13.) $\int_{\partial R} P\,dx + Q\,dy = \iint_R Q_x - P_y \, dA = \iint_R 3\, dA$

$= 3 \text{ Area}(R) = 3(\pi \cdot 2^2) = \boxed{12\pi}$

since R is a disk of radius 2.

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(14.) $z = 2x + 4y + 1$ and $x^2+y^2 \le 1$.

$\vec{F} = (y, y, 0)$

(a) $\vec{n}\,dS = (-2,-4,1)\,dx\,dy$ so

$dS = \sqrt{4+16+1}\,dx\,dy = \sqrt{21}\,dx\,dy$

Then $\text{Surface Area}(S) = \iint_S dS = \iint_{x^2+y^2\le 1} \sqrt{21}\,dx\,dy = \boxed{\sqrt{21}\cdot\pi}$

(b) $\nabla\times\vec{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y & y & 0 \end{vmatrix} = (0,0,-1)$ so

$\iint_S \nabla\times\vec{F}\cdot\vec{n}\,dS = \iint (0,0,-1)\cdot(-2,-4,1)\,dx\,dy = -\iint dx\,dy$

$\boxed{= -\pi}$

(c) $\int_{\partial S} \vec{F}\cdot d\vec{r} = \iint_S \nabla\times\vec{F}\cdot\vec{n}\,dS = \boxed{-\pi}$ also