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Math 2020, Winter 2007, Quiz 11
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Determine whether or not the following series converge absolutely, converge conditionally, or diverge, and explain how you know.

(1) $\sum(-1)^n \frac{\sqrt{n}}{n+1}$ (2) $\sum(-1)^n \frac{3^n}{2^n n^2}$ (3) $\sum(-1)^n \frac{2^n}{n^n}$

(1) $\frac{\sqrt{n}}{n+1} < \frac{\sqrt{n}}{n} = \frac{1}{\sqrt{n}}$ and $\sum \frac{1}{\sqrt{n}}$ is a divergent p-series ($p = 1/2$). Since

$$\frac{\frac{\sqrt{n}}{n+1}}{\frac{1}{\sqrt{n}}} = \frac{n}{n+1} \to 1$$

both series behave the same, so $\sum (-1)^n \frac{\sqrt{n}}{n+1}$ does not converge absolutely. Now, terms decrease,

i.e. $\frac{\sqrt{n+1}}{(n+1)+1} < \frac{\sqrt{n}}{n+1}$

iff $(n+1)^{3/2} < \sqrt{n}\,(n+2)$

iff $(n+1)^3 < n\,(n+2)^2$

iff $n^3 + 3n^2 + 3n + 1 < n^3 + 4n^2 + 4n$

iff $1 < n^2 + n$

and this is true so $\sum (-1)^n \frac{\sqrt{n}}{n+1}$ converges conditionally by the alternating series test.

(2) Ratio test:
$$\frac{\dfrac{3^{n+1}}{2^{n+1}(n+1)^2}}{\dfrac{3^n}{2^n n^2}} = \frac{3^{n+1}}{3^n}\,\frac{2^n}{2^{n+1}}\left(\frac{n}{n+1}\right)^2$$

$$\longrightarrow \frac{3}{2} * 1 = \frac{3}{2}$$

so this diverges by the ratio test.

(3) Ratio test:
$$\frac{\dfrac{2^{n+1}}{(n+1)^{n+1}}}{\dfrac{2^n}{n^n}}$$
Yuck!

Root test:
$$\sqrt[n]{\frac{2^n}{n^n}} = \frac{2}{n} \longrightarrow 0$$

so this converges absolutely