Sample Final Exam problems — Solutions

Quick & dirty version. Let me know if you spot any errors.
ZB

1. (a) $\int_0^5 f(x)\,dx = \int_0^2 f(x)\,dx + \int_2^5 f(x)\,dx = 5 + 11 = \boxed{16}$

(b) $\int_0^2 2f(x) - 3g(x)\,dx = 2\int_0^2 f(x)\,dx - 3\int_0^2 g(x)\,dx = 2(5) - 3(7) = \boxed{-11}$

2. Avg of x^2 on $[0,2]$ $= \frac{1}{2}\int_0^2 x^2\,dx = \frac{1}{6}x^3\Big|_0^2 = \frac{8}{6} = \boxed{\frac{4}{3}}$

3. (a) $\frac{d}{dx}\int_1^x \ln(1+\ln t)\,dt = \boxed{\ln(1+\ln x)}$

(b) $\frac{d}{dx}\int_1^2 x\ln(1+\ln t)\,dt = \boxed{\int_1^2 \ln(1+\ln t)\,dt}$ if x is independent of t.

(c) $\frac{d}{dx}\int_x^4 \ln(1+\ln t)\,dt = \boxed{-\ln(1+\ln x)}$

(d) $\frac{d}{dx}\int_x^{x^2} \ln(1+\ln t)\,dt = \boxed{\ln(1+\ln x^2) * 2x - \ln(1+\ln x)}$

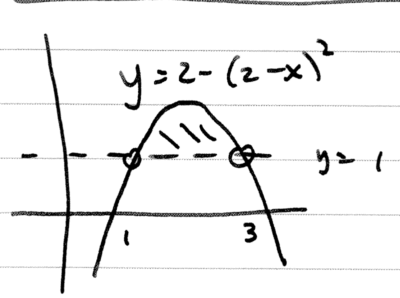
4. $\Delta x = \frac{3-1}{3} = \frac{2}{3}$

$[1, 1\frac{2}{3}]$ $\quad \frac{2}{3} * \frac{1}{4/3} = \frac{2}{3} * \frac{3}{4} = \frac{1}{2}$

$[1\frac{2}{3}, 2\frac{1}{3}]$ $\quad \frac{2}{3} * \frac{1}{2} = \frac{1}{3}$

$[2\frac{1}{3}, 3]$ $\quad \frac{2}{3} * \frac{1}{8/3} = \frac{2}{3} * \frac{3}{8} = \frac{1}{4}$

$$\boxed{\int_1^3 \frac{dx}{x} \simeq \frac{1}{2} + \frac{1}{3} + \frac{1}{4}}$$

5. 

$\text{Vol} = \pi\int_1^3 r_{outer}^2 - r_{inner}^2\,dx$

$= \pi\int_1^3 (2-(x-2)^2)^2 - 1\,dx$

$= \pi\int_1^3 (x-2)^4 - 4(x-2)^2 + 3\,dx$

(CONT.)

5. (cont.) $= \pi \int_{-1}^{1} x^4 - 4x^2 + 3 \, dx = \pi \left[\frac{x^5}{5} - \frac{4x^3}{3} + 3x \Big|_{-1}^{1} \right]$

$= 2\pi \left(\frac{1}{5} - \frac{4}{3} + 3 \right) = \frac{2\pi}{15} \left(3 - 20 + 45 \right) = \boxed{\frac{56\pi}{15}}$

6. (a) $\frac{1}{x^2 - x - 2} = \frac{1}{(x+1)(x-2)} = \frac{A}{x+1} + \frac{B}{x-2}$

$1 = A(x-2) + B(x+1)$

$x = 2$ $\quad 1 = A \cdot 0 + 3B \qquad B = 1/3$

$x = -1$ $\quad 1 = -3A + 0 \qquad A = -1/3$

$\int \frac{dx}{x^2 - x - 2} = \frac{1}{3} \int \frac{-1}{x+1} + \frac{1}{x-2} \, dx = \boxed{\frac{1}{3}\left(-\ln|x+1| + \ln|x+2|\right) + C}$

(b) $\int \frac{x^2 + 2x}{x^2 + 4} \, dx = \int 1 + \frac{2x}{x^2+4} - \frac{4}{x^2+4} \, dx$ $\qquad u = x^2 + 4$, $du = 2x\,dx$

$= \boxed{x + \ln(x^2+4) - 2\tan^{-1}\left(\frac{x}{2}\right) + C}$

since

$\int \frac{dx}{x^2+4} = \frac{1}{4} \int \frac{dx}{\left(\frac{x}{2}\right)^2 + 1} = \frac{1}{2} \int \frac{d(x/2)}{(x/2)^2 + 1} = \frac{1}{2} \tan^{-1}\left(\frac{x}{2}\right)$

(c) $\int x \sin x \, dx = -x\cos x + \int -\cos x \, dx = \boxed{-x\cos x - \sin x + C}$

$u = x \qquad du = dx$

$dv = \sin x \, dx \qquad v = -\cos x$

(d) $\int \sin^{10} x \cos x \, dx = \int u^{10} du = \frac{1}{11} u^{11} + c = \boxed{\frac{1}{11} \sin^{11} x + C}$

$u = \sin x$
$du = \cos x \, dx$

(e) $\int_0^\infty \frac{dx}{\sqrt{x}} = \lim_{A \to \infty} \int_0^A x^{-1/2} dx = \lim_{A \to \infty} 2\sqrt{A} = \boxed{\infty}$

Really we also need $\lim_{\varepsilon \to 0^+} -2\sqrt{\varepsilon} = 0$ but adding 0 has no effect.

7. $x = t^3 - 3t$, $y = t^4 - 3t^2$

(a) $\boxed{\frac{dx}{dt} = 3t^2 - 3}$ $\boxed{\frac{dy}{dt} = 4t^3 - 6t}$

(b) Vertical $\Leftrightarrow dx/dt = 0 \Leftrightarrow t^2 = 1 \Leftrightarrow \boxed{t = \pm 1}$

(c) Horizontal $\Leftrightarrow dy/dt = 0 \Leftrightarrow 2t(2t^2 - 3) = 0 \Leftrightarrow \boxed{t = 0, t = \sqrt{3/2}, t = -\sqrt{3/2}}$

(d) At $t = 1/\sqrt{3}$, $x = \frac{1}{\sqrt{3}}\left(\frac{1}{3} - 3\right) = -8/3\sqrt{3}$, $x' = -2$

$y = \frac{1}{3}\left(\frac{1}{3} - 3\right) = -8/9$ $y' = \frac{2}{\sqrt{3}}\left(\frac{2}{3} - 3\right) = \frac{-14}{3\sqrt{3}}$

$$\boxed{\text{Tangent} \quad \left(\frac{-8}{3\sqrt{3}}, \frac{-8}{9}\right) + t\left(-2, \frac{-14}{3\sqrt{3}}\right)}$$

(e) Arclength: $\int_0^{1/\sqrt{3}} \sqrt{(x')^2 + (y')^2} \, dt =$

$$\boxed{= \int_0^{1/\sqrt{3}} \sqrt{(3t^2 - 3)^2 + (4t^3 - 6t)^2} \, dt}$$

Series part

1. (a) Alt. $(-1)^n$, Decr: $\frac{(n+1)^2+1}{(n+1)^3+2} < \frac{n^2+1}{n^3+2}$ for large n since they are $\approx \frac{1}{n+1} < \frac{1}{n}$

Term $\to 0$ $\left(\frac{n^2+1}{n^3+2} = \frac{1+1/n}{n+2/n}\right)$ so conv. by alt. series test.

Since $\frac{n^2+1}{n^3+2} \Big/ \frac{1}{n} \to 1$, it does not conv. abs.

So $\boxed{\text{conv. conditionally}}$.

(b) $|-2/3| < 1$ so $\boxed{\text{conv. abs.}}$ (geom w/ $r = -2/3$)

(c) $\frac{2^n+n}{3^n+1} \Big/ \left(\frac{2}{3}\right)^n \to 1$ so $\boxed{\text{conv. abs.}}$ by comparison w/ geom series w/ $r = 2/3$

(d) Ratio: $\frac{2/(n+1)!}{2/n!} = \frac{1}{n+1} \to 0$ so $\boxed{\text{conv. abs}}$

(e) $\frac{n^2+1}{n^2+3} \to 1$ so $\boxed{\text{diverges}}$ by n^{th} term test.

2. Conv. abs. if $\left| \frac{5^{n+1} x^{2n+2}}{(n+1)^2} \frac{n^2}{5^n x^{2n}} \right| < 1$

i.e. if $5x^2 < 1$ so Radius of Conv $= \boxed{1/\sqrt{5}}$

3. $\lim_{x \to 0} \frac{x(1-\cos x)}{x - \sin x} = \lim_{x \to 0} \frac{-x^3/2 + \text{higher}}{x^3/6 + \text{higher}} = -\frac{6}{2} = \boxed{-3}$

4. (a) $\boxed{\sum_{n=1}^{\infty} \frac{x^{n+1}}{n^2}}$ (b) $\boxed{\sum_{1}^{\infty} \frac{x^{n-1}}{n}}$

(5.) $\sin(0.6) \approx 0.6 - \frac{(0.6)^3}{6} + \frac{(0.6)^5}{5!} - \cdots$ is an alternating series, so the 2 term approximation

$$0.6 - \frac{(0.6)^3}{6} = 0.6 - (0.1)(0.36) = 0.564$$

is accurate to within $< \frac{(0.6)^5}{5!} = \frac{6^5 * 10^5}{10 * 12} = 3 * 6^3 * 10^{-6} = .000648$

Precisely $\boxed{0.564 < \sin(0.6) < 0.564648}$

(6.) Recall that with an alternating, decreasing series, the error

$$\sum_1^\infty f(n) - \sum_1^N f(n) = \sum_{N+1}^\infty f(n)$$

is trapped between two integrals:

$$\int_{N+1}^\infty f(x)\,dx \le \sum_{N+1}^\infty f(n) < \int_N^\infty f(x)\,dx$$

So with $N=3$ we get $\sum_1^3 \frac{n}{(n^2+1)^2} = \frac{1}{4} + \frac{2}{25} + \frac{3}{100} = \frac{36}{100}$

with error between

$$\int_4^\infty \frac{x}{(x^2+1)^2}\,dx = \frac{1}{2}\int_{17}^\infty \frac{du}{u^2} = \frac{-1}{2u}\Big|_{17}^\infty = \frac{1}{34}$$

$u = x^2+1$

$\frac{1}{2}du = x\,dx$

and $\int_3^\infty \frac{x}{(x^2+1)^2}\,dx = \frac{-1}{2u}\Big|_{10}^\infty = \frac{1}{20}$

So $\boxed{\frac{36}{100} + \frac{1}{34} < \sum_1^\infty \frac{n}{(n^2+1)^2} < \frac{36}{100} + \frac{1}{20} = \frac{41}{100}}$

$\approx .3894$

The average of these, $\boxed{.3997}$, is then accurate to within $\boxed{.0103}$

7\. (a) $\frac{1+i}{1-i} = \frac{(1+i)^2}{2} = \frac{2i}{2} = \boxed{i}$

(b) $|2+3i| = \sqrt{4+9} = \boxed{\sqrt{13}}$

(c) $\left(\frac{1+i}{\sqrt{2}}\right)^{103} = \left(e^{\pi i/4}\right)^{103} = e^{\frac{103}{4}\pi i} = e^{24\pi i + \frac{7}{4}\pi i} = e^{\frac{7}{4}\pi i}$

$= \cos\left(\frac{7}{4}\pi\right) + i \sin\left(\frac{7}{4}\pi\right) = \boxed{\frac{1-i}{\sqrt{2}}}$

(d) $\sqrt[3]{\frac{-1+i}{\sqrt{2}}} = \sqrt[3]{e^{3\pi i/4}} = e^{\pi i/4} = \boxed{\frac{1+i}{\sqrt{2}}}$

The other two are $\boxed{\left(\frac{1+i}{\sqrt{2}}\right)\left(\frac{-1+i\sqrt{3}}{2}\right)}$ and $\boxed{\left(\frac{1+i}{\sqrt{2}}\right)\left(\frac{-1-i\sqrt{3}}{2}\right)}$.

8\. $f(x) = \sum \frac{n+1}{n^2+1} x^n = \sum \frac{f^{(n)}(0)}{n!} x^n$ so

$\frac{f^{(n)}(0)}{n!} = \frac{n+1}{n^2+1}$ so $f^{(n)}(0) = (n+1)!/(n^2+1)$

so $\boxed{f^{(7)}(0) = \frac{8!}{50}} = \frac{8\cdot 7\cdot 6\cdot 5\cdot 3\cdot 2}{50} = \frac{8\cdot 7\cdot 6\cdot 3}{5}$

9\. Taylor series $= \sum_0^{\infty} \frac{f^{(n)}(0)}{n!} x^n$

$$\boxed{= \sum_{n=0}^{\infty} \frac{n^2+1}{n!} x^n}$$