Final Exam, Math 2020
25 April 2007
R. Bruner

Write clearly, label your answers, and leave space between problems.
I strongly suggest that you read all the problems before starting work.

1. (5) Compute $\dfrac{d}{dx}\displaystyle\int_{x^2}^{x^3} \sqrt{\sin(t)}\, dt$

   Sol: $\sqrt{\sin(x^3)} \cdot 3x^2 - \sqrt{\sin(x^2)} \cdot 2x$

2. (5) Compute $\dfrac{d}{dx}\displaystyle\int \sqrt{\sin(x)}\, dx$

   Sol: $\sqrt{\sin(x)}$

3. (10) Find the average value of $\cos(x)$ on the interval $[0, \pi/2]$.

   Sol:

$$\frac{1}{\pi/2}\int_0^{\pi/2} \cos(x)\, dx = \frac{2}{\pi}(\sin(\pi/2) - \sin(0)) = 2/\pi$$

4. (10) Find the area of the region below $y = 3x - x^2$ and above $y = x$.

   Sol: $x < 3x - x^2$ when $x^2 - 3x + x < 0$, i.e., $x^2 - 2x < 0$. Since $x^2 - 2x = x(x-2)$, this is $0 \le x \le 2$. The area is therefore

$$\int_0^2 2x - x^2\, dx = (x^2 - x^3/3)\Big|_0^2 = 4 - 8/3 = 4/3$$

5. (10) The region in problem (4) is revolved about the x-axis. Find the volume of the resulting solid.

   Sol:

$$\pi\int_0^2 r_{\text{outer}}^2 - r_{\text{inner}}^2\, dx = \pi\int_0^2 (3x - x^2)^2 - x^2\, dx = \pi\int_0^2 9x^2 - 6x^3 + x^4 - x^2\, dx =$$

$$\pi(\frac{8}{3}x^3 - \frac{6}{4}x^4 + \frac{1}{5}x^5)\Big|_0^2 = \pi(\frac{64}{3} - 24 + \frac{32}{5}) = \frac{56}{15}$$

6. (10) Compute $\int 2xe^{3x}\,dx$

Sol: Use parts with $u = 2x$ and $dv = e^{3x}dx$ or use undetermined coefficients:

$$\frac{d}{dx}(ax+b)e^{3x} = ae^{3x} + 3(ax+b)e^{3x} = (3ax + (a+3b))e^{3x}$$

so $a + 3b = 0$ and $3a = 2$. Thus $a = (2/3)$ and $b = -2/9$. Hence

$$\int 2xe^{3x}\,dx = \left(\frac{2}{3}x - \frac{2}{9}\right)e^{3x} + C.$$

7. (10) Compute $\int (x+1)(x+4)^{99}\,dx$

Sol: Let $u = x + 4$. Then $x = u - 4$ and $dx = du$ so

$$\int (x+1)(x+4)^{99}\,dx = \int (u-3)u^{99}\,du = \int u^{100} - 3u^{99}\,du$$

$$= \frac{1}{101}u^{101} - \frac{3}{100}u^{100} + C = \frac{1}{101}(x+4)^{101} - \frac{3}{100}(x+4)^{100} + C$$

8. (10) Compute $\int x^3 \ln x\,dx$

Sol: Use integration by parts with $u = \ln x$ and $dv = x^3\,dx$. Then $du = (1/x)\,dx$ and $v = x^4/4$ gives

$$\int x^3 \ln x\,dx = uv - \int v\,du = \frac{x^4 \ln x}{4} - \int \frac{x^3}{4}\,dx = \frac{x^4 \ln x}{4} - \frac{x^4}{16} + C$$

9. (10) Compute $\int \frac{1}{(x+1)(x+4)}\,dx$

Sol: The partial fraction decomposition

$$\frac{1}{(x+1)(x+4)} = \frac{A}{x+1} + \frac{B}{x+4}$$

requires that $1 = A(x+4) + B(x+1)$. Therefore $A + B = 0$ and $4A + B = 1$. Hence $B = -A$, so $3A = 1$ and $A = 1/3$, $B = -1/3$. Therefore

$$\int \frac{1}{(x+1)(x+4)}\,dx = \frac{1}{3}\int \frac{1}{x+1} - \frac{1}{x+4}\,dx = \frac{1}{3}\left(\ln(x+1) - \ln(x+4)\right) + C$$

$$= \frac{1}{3}\ln\left(\frac{x+1}{x+4}\right) + C = \frac{1}{3}\ln(1 - \frac{3}{x+4}) + C$$

10. (10) Compute $\int \frac{1}{\sqrt{4-9x^2}}\,dx$

Sol:

$$\int \frac{1}{\sqrt{4-9x^2}}\,dx = \frac{1}{2}\int \frac{1}{\sqrt{1-9x^2/4}}\,dx$$

Let $u = (3/2)x$, so that $dx = (2/3)du$ and $1 - 9x^2/4 = 1 - u^2$. Then the integral becomes

$$\left(\frac{1}{2}\right)\left(\frac{2}{3}\right)\int \frac{1}{\sqrt{1-u^2}}\,du = \frac{1}{3}\sin^{-1}(u) + C = \frac{1}{3}\sin^{-1}(3x/2) + C$$

11. (10) Compute $\int \frac{x}{\sqrt{4-9x^2}}\,dx$

Sol: Let $u = 4 - 9x^2$, so that $du = -18x\,dx$, and hence, $x\,dx = -(1/18)\,du$. Then

$$\int \frac{x}{\sqrt{4-9x^2}}\,dx = \frac{-1}{18}\int \frac{du}{\sqrt{u}} = \frac{-1}{18}\int u^{-1/2}\,du = \frac{-1}{18}(2u^{1/2}) + C = \frac{-1}{9}\sqrt{4-9x^2} + C$$

12. (10) Compute $\int_1^\infty \frac{1}{x^4}\,dx$

Sol:

$$\int_1^\infty \frac{1}{x^4}\,dx = \lim_{A\to\infty}\int_1^A \frac{1}{x^4}\,dx = \lim_{A\to\infty} -\frac{1}{3}\frac{1}{x^3}\Big|_1^A = \lim_{A\to\infty}\left(\frac{1}{3} - \frac{1}{3A^3}\right) = \frac{1}{3}$$

13. (15) Estimate $\int_0^4 \frac{16}{x^2+16}\,dx$ using a Riemann sum with two equal subintervals and

(a) left hand endpoints

(b) right hand endpoints

What do these estimates tell you about π?

Sol: (a): The left endpoint approximation uses $x_1^* = 0$ on $[0,2]$ and uses $x_2^* = 2$ on $[2,4]$. This gives

$$\frac{16}{0^2+16}(2-0) + \frac{16}{2^2+16}(4-2) = 2 + \frac{4}{5}(2) = 3\frac{3}{5}.$$

(b): The right endpoint approximation uses $x_1^* = 2$ on $[0,2]$ and uses $x_2^* = 4$ on $[2,4]$. This gives

$$\frac{16}{2^2+16}(2-0) + \frac{16}{4^2+16}(4-2) = \frac{4}{5}(2) + 1 = 2\frac{3}{5}.$$

Since the function is decreasing on $[0,4]$ and the integral equals π, we have

$$2\frac{3}{5} < \pi < 3\frac{3}{5}.$$

14. Consider the curve $x(t) = t^3 + t,\ y(t) = t^4 - 2t^2$.

(a) (10) Compute dx/dt and dy/dt.

(b) (5) Compute dy/dx.

(c) (5) Show that the curve is never vertical.

(d) (5) Find all three places where the curve is horizontal.

(e) (10) Compute the area between the curve $(x(t), y(t))$, $0 \le t \le 1$, and the x-axis.

(f) (5) Write the integral which computes the length of the curve from $t = 0$ to $t = 1$.

Sol: (a) $dx/dt = 3t^2 + 1$ and $dy/dt = 4t^3 - 4t$.
(b) $dy/dx = (4t^3 - 4t)/(3t^2 + 1)$.
(c) The curve is vertical iff $dx/dt = 0$ but $3t^2 + 1 \ge 1 > 0$ for all t.
(d) The curve is horizontal iff $dy/dt = 0$, i.e. $4t^3 - 4t = 0$. Since $t^3 - t = t(t-1)(t+1)$ this happens when $t = -1, 0, 1$. These are the points $(x, y) = (-2, -1)$, $(0, 0)$ and $(2, -1)$.
(e)

$$\int_0^1 y\,dx = \int_0^1 (t^4 - 2t^2)(3t^2 + 1)\,dt = \int_0^1 3t^6 - 5t^4 - 2t^2\,dt = \frac{3}{7} - 1 - \frac{2}{3} = -\frac{26}{21}$$

The value of $y(t)$ is negative throughout $[0, 1]$, so this integral is the negtive of the area. Hence the area is $26/21$.
(f)

$$\int_0^1 \sqrt{(3t^2 + 1)^2 + (4t^3 - 4t)^2}\,dt$$

15. (a) (5) Convert the polar curve $r^2 = \cos 2\theta$ to Cartesian coordinates. (Hint: $\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$.)

(b) (10) The part with $r \ge 0$ and $-\pi/4 \le \theta \le \pi/4$ forms a loop centered on the x-axis. Use the polar form to compute the area inside this loop.

Sol: (a) Multiplying by r^2 we find the curve is given by

$$r^4 = r^2 \cos(2\theta) = r^2 \cos^2\theta - r^2 \sin^2\theta.$$

In Cartesian coordinates this becomes

$$(x^2 + y^2)^2 = x^2 - y^2$$

(b)

$$\int_{-\pi/4}^{\pi/4} \frac{1}{2} r^2\,d\theta = \int_{-\pi/4}^{\pi/4} \frac{1}{2} \cos 2\theta\,d\theta = \frac{1}{4} \sin 2\theta \Big|_{-\pi/4}^{\pi/4} = \frac{1}{4} - \left(-\frac{1}{4}\right) = \frac{1}{2}$$

16. (5) Convert $xy = 1$ to polar coordinates .

    Sol: $r\cos(\theta) r\sin(\theta) = 1$ or $r^2 \sin(\theta)\cos(\theta)$

17. (10) Compute $\lim_{x\to 0} \dfrac{1-\cos(x) - x\sin(x)}{e^x - 1 - x}$

    Sol:

$$\frac{1 - \left(1 - \dfrac{x^2}{2} + \cdots\right) - \left(x - \dfrac{x^3}{6} + \cdots\right)}{\left(1 + x + \dfrac{x^2}{2} + \cdots\right) - 1 - x} = \frac{-\dfrac{x^2}{2} + \text{higher}}{\dfrac{x^2}{2} + \text{higher}} \longrightarrow -1$$

18. (10) Find the Taylor series for $(e^x - 1)/x$.

    Sol:

$$\frac{1}{x}\sum_{n=1}^{\infty} \frac{x^n}{n!} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{n!} = \sum_{n=0}^{\infty} \frac{x^n}{(n+1)!}$$

19. (10) Use the Taylor series of $\cos(x)$ to show that the approximation $\cos(1/2) \simeq 1 - (1/8) = 7/8$ is accurate to two decimal places.

    Sol: The Taylor series for $\cos x$ is an alternating series, so the error will be less than the first omitted term, which is $(1/2)^4/4! = 1/(16 * 24) < 1/100$.

20. (10) Find a power series for $\int e^{-x^3}\, dx$

    Sol:

$$\int e^{-x^3}\, dx = \int \sum_0^{\infty} \frac{(-x^3)^n}{n!}\, dx = \int \sum_0^{\infty} \frac{(-1)^n x^{3n}}{n!}\, dx = \sum_0^{\infty} (-1)^n \frac{x^{3n+1}}{n!(3n+1)}$$

21. (10 each) Determine whether the following series converge absolutely, converge conditionally, or diverge:

    (a) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$ Sol: converges conditionally: convergent alt series, but absolute values diverge by comparison test with divergent harmonic series.

    (b) $\displaystyle\sum_{n=0}^{\infty} \frac{n+1}{2n+1}$ Sol: diverges by n^{th} term test

    (c) $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{n^2}{2^n}$ Sol: converges absolutely by ratio test:

$$\frac{(n+1)^2}{2^{n+1}} \frac{2^n}{n^2} = \left(\frac{n+1}{n}\right)^2 \cdot \frac{1}{2} \longrightarrow \frac{1}{2} < 1$$

22. (10) Find the radius of convergence of $\sum_{n=1}^{\infty} \frac{2^n x^n}{n^2}$

Sol: $R = 1/2$ by the ratio test:

$$\frac{2^{n+1}x^{n+1}}{(n+1)^2} \frac{n^2}{2^n x^n} = 2x \left(\frac{n}{n+1} \right)^2 \longrightarrow 2x < 1$$

iff $x < 1/2$.

23. (10) If $f(x) = \sum_{n=1}^{\infty} \frac{2^n x^n}{n^2}$, what is $f^{(n)}(0)$?

Sol: $\frac{f^{(n)}(0)}{n!} = \frac{2^n}{n^2}$ so $f^{(n)}(0) = \frac{2^n n!}{n^2}$

24. (5 each) Compute

$$(a)\ \frac{2+i}{1+i} \qquad (b)\ \sqrt{i} \qquad (c)\ |3-4i| \qquad (d)\ i^{103}$$

Sol: (a) $\frac{2+i}{1+i} = \frac{(2+i)(1-i)}{(1+i)(1-i)} = \frac{2-2i+i+1}{2} = \frac{3-i}{2}$

(b) $\sqrt{i} = \sqrt{e^{\pi i/2}} = \pm e^{\pi i/4} = \pm(1+i)/2$

(c) $|3-4i| = \sqrt{3^2+4^2} = \sqrt{9+16} = \sqrt{25} = 5$

(d) $i^{103} = i^{100} i^3 = (i^4)^{25} i^3 = 1 \cdot i^3 = -i$

——— The End ———