(2007-04-04 15:35)

THE SEMI-DIHEDRAL ALGEBRA IN ALGEBRAIC TOPOLOGY

ROBERT R. BRUNER

ABSTRACT. We explain how the 8 dimensional semi-dihedral algebra is of interest to algebraic topologists.

The semi-dihedral algebra of dimension 8 over $\mathbf{F}_2$ arises when considering real connective K-theory. Since real K-theory is the C_2-fixed part of complex K-theory (either periodic or connective), the difference between real and complex is insignificant at primes other than 2. Hence we shall focus attention purely on the 2-primary situation.

We start by giving a little background on connective K-theory to explain the context, and then describe where the semi-dihedral algebra comes in.

Let $H = H\mathbf{F}_2$ and $H\mathbf{Z}$ be the spectra (in the sense of stable homotopy theory) representing ordinary mod 2 cohomology and ordinary integral cohomology, respectively. Let ko and ku be the spectra representing connective real and complex K-theory, respectively, and let KO and KU be the spectra representing periodic real and complex K-theory. Thus, H and $H\mathbf{Z}$ are determined by simplices, e.g., via singular chains and cochains of a space, while KO and KU are determined by real and complex vector bundles, e.g., on a space X, respectively. (The value of$KU^*(X)$ when X is a spectrum, rather than a space, is a more homotopy theoretical object, not quite as directly related to vector bundles.)

The connective theories, ko and ku, mix KO and KU with HZ. There are a couple of ways to look at this. First, we have cofiber sequences exposing the relation to cohomology:

$$\Sigma ko \xrightarrow{\eta} ko \xrightarrow{c} ku \xrightarrow{R} \Sigma^2 ko$$

$$\Sigma^2 ku \xrightarrow{v} ku \longrightarrow H\mathbf{Z} \longrightarrow \Sigma^3 ku$$

$$H\mathbf{Z} \xrightarrow{2} H\mathbf{Z} \longrightarrow H\mathbf{F}_2 \longrightarrow \Sigma H\mathbf{Z}$$

The latter sequence is purely arithmetic, of course, but is included to show the analogy between ku's relation to $H\mathbf{Z}$ and $H\mathbf{Z}$'s better known relation to $H\mathbf{F}_2$. Here v is the map representing complex Bott periodicity, c is complexification, and R is a lift of realification, in that $\Sigma^2 ku \xrightarrow{v} ku \xrightarrow{R} \Sigma^2 ko$ is the double suspension of realification. The map η is related to the fact that $ko = (ku)^{C_2}$, the part of complex K-theory fixed by complex conjugation. Second, the periodic theories are obtained from the connective ones by inverting Bott periodicity:

$$KU = ku[1/v] = \varinjlim(ku \xrightarrow{v} \Sigma^{-2} ku \xrightarrow{v} \Sigma^{-4} ku \xrightarrow{v} \cdots)$$

and similarly $KO = ko[1/v^4]$. This latter makes sense because the map of coefficients $ko_* \xrightarrow{c} ku_*$ hits the subalgebra generated by v^4 (and more). So, roughly speaking, ku modulo v is cohomology, $ku/v = H\mathbf{Z}$, while ku with v inverted is bundle theory, $ku[1/v] = KU$

On the level of coefficient rings this is especially clear. Recall that the coefficient ring of a ring spectrum is its homotopy, $E_i = \pi_i(E)$, with product induced by the ring structure $E \wedge E \longrightarrow E$. Here, we have

- $H\mathbf{F}_{2*} = \mathbf{F}_2$
- $H\mathbf{Z}_* = \mathbf{Z}$
- $ku_* = \mathbf{Z}[v]$
- $KU_* = Z[v, v^{-1}]$
- $ko_* = Z[\eta, \alpha, \beta]/(2\eta, \eta^3, \eta\alpha, \alpha^2 - 4\beta)$, and
- $KO_* = ko_*[\beta^{-1}]$

Thus $Spec(ku_*)$ is composed of the open set $\{v \neq 0\} = Spec(KU_*)$, and its complement, the closed set $\{v = 0\} = Spec(H\mathbf{Z}_*)$. Similary, $Spec(ku^*(X))$ is augmented over $Spec(ku_*)$ for any space X, and the parts sitting over $\{v = 0\}$ and $\{v \neq 0\}$ are given by the cohomology and the periodic K-theory of X respectively.

1991 *Mathematics Subject Classification.* Primary: ; Secondary:

The coefficients for real K-theory are more complex, and I will only note here that η induces the map η mentioned in the relation between ko and ku, and that complexification induces the ring homomorphism $c_* : ko_* \longrightarrow ku_*$ which satisfies $c_*(\eta) = 0$, $c_*(\alpha) = 2v$, and $c_*(\beta) = v^4$. From this and degree considerations, the relation $\alpha^2 = 4\beta$ is obvious. The most algebraic derivation of these facts is by considering real and complex Clifford algebras, as in [2] and [1]. The difference between ko and ku is thus not visible on the level of varieties, since c_* is an F-isomorphism.

The relation to the semi-dihedral algebra arises when considering the relation between mod 2 cohomology and real connective K-theory. We shall simultaneously describe the relation of mod 2 cohomology to integral cohomology and complex K-theory, to provide context.

Let $\mathcal{A}$ be the mod 2 Steenrod algebra, i.e., $\mathrm{End}(H\mathbf{F}_2)$, the algebra of natural transformations of mod 2 cohomology. The Steenrod operations Sq^i, $i \geq 0$, generate $\mathcal{A}$ subject to the Adem relations, such as $Sq^0 = 1$, $Sq^1Sq^1 = 0$, $Sq^2Sq^2 = Sq^1Sq^2Sq^1$, etc. The Steenrod algebra is a Hopf algebra with coproduct $\psi(Sq^n) = \sum_{i+j=n} Sq^i \otimes Sq^j$. When we say algebra or subalgebra from now on, we shall mean Hopf algebra or subalgebra.

The subalgebra $\mathcal{A}(n)$ is generated by Sq^i for $i \leq 2^n$. Thus $\mathcal{A}(0)$ is the exterior algebra generated by Sq^1, while $\mathcal{A}(1)$, which is generated by Sq^1 and Sq^2 subject to the two relations mentioned above, is the semi-dihedral algebra which has dimension 8 over $\mathbf{F}_2$. In this topological setting, $\mathcal{A}$ and $\mathcal{A}(n)$ are graded by $\deg(Sq^i) = i$. Thus, for example, the socle of $\mathcal{A}(1)$ is $\Sigma^6 F_2 = \mathbf{F}_2[6]$, i.e., F_2 concentrated in degree 6, and is generated by $Sq^2Sq^2Sq^2 = (Sq^1Sq^2)^2 = (Sq^2Sq^1)^2$. The Hilbert series of $\mathcal{A}(1)$ is $(1+t)(1+t^2)(1+t^3)$.

Following Milnor ([5]), we define $Q_0 = Sq^1$ and $Q_n = [Q_{n-1}, Sq^{2^n}]$ for $n > 0$. Let $E(n)$ be the subalgebra of $\mathcal{A}$ generated by $Q_0, \ldots, Q_n$, which Milnor shows is an exterior algebra. Clearly $E(n)$ is a subalgebra of $\mathcal{A}(n)$. In particular, $E(0) = \mathcal{A}(0)$ while $E(1)$ has index 2 in $\mathcal{A}(1)$.

Now, it turns out that

$$H^*(H\mathbf{Z}) = \mathcal{A} \otimes_{E(0)} \mathbf{F}_2,$$

$$H^*(ku) = \mathcal{A} \otimes_{E(1)} \mathbf{F}_2,$$

and

$$H^*(ko) = \mathcal{A} \otimes_{\mathcal{A}(1)} \mathbf{F}_2$$

Therefore, by the Künneth Theorem and a standard lemma about extended modules over Hopf algebras, we have

$$H^*(H\mathbf{Z} \wedge X) = \mathcal{A} \otimes_{E(0)} H^*X,$$

$$H^*(ku \wedge X) = \mathcal{A} \otimes_{E(1)} H^*X,$$

and

$$H^*(ko \wedge X) = \mathcal{A} \otimes_{\mathcal{A}(1)} H^*X$$

It follows that the mod 2 cohomology of $E \wedge X$ as a $\mathcal{A}$-module, where E is $H\mathbf{Z}$, ku, or ko, is entirely determined by the mod 2 cohomology of X as a module over $E(0)$, $E(1)$, and $\mathcal{A}(1)$, respectively. This is significant because, by definition, $E_*(X) = \pi_*(E \wedge X)$ and $E^*(X) = \pi_* F(X, E) = \pi_*(F_E(E \wedge X, E))$, where $F(-,-)$ and $F_E(-,-)$ are the function spectra of all maps and of E-module maps, respectively. Thus, information about $E \wedge X$ is relevant to computing the E-homology and cohomology of X.

Here is a way to understand the relation between the theories $H\mathbf{Z}$, ku and ko and the subalgebras of $\mathcal{A}$ to which they correspond. The cofiber sequence $H\mathbf{Z} \longrightarrow H\mathbf{Z} \longrightarrow H \longrightarrow \Sigma H\mathbf{Z}$ gives rise to an exact couple for computing $H\mathbf{Z}^*X$ from H^*X whose first differential is $H \longrightarrow \Sigma H\mathbf{Z} \longrightarrow \Sigma H$, and this turns out to be Sq^1, which generates $E(0)$. The cofiber sequence $\Sigma^2 ku \longrightarrow ku \longrightarrow H\mathbf{Z} \longrightarrow \Sigma^3 ku$ gives rise to an exact couple for computing ku^*X from $H\mathbf{Z}^*X$ whose first differential is $H\mathbf{Z} \longrightarrow \Sigma^3 ku \longrightarrow \Sigma^3 H\mathbf{Z}$, and this turns out to be a lift to integral cohomology of Q_1, which, together with $E(0)$, generates $E(1)$. Finally, the cofiber sequence $\Sigma ko \longrightarrow ko \longrightarrow ku \longrightarrow \Sigma^2 ko$ gives rise to an exact couple for computing ko^*X from ku^*X whose first differential is $ku \longrightarrow \Sigma^2 ko \longrightarrow \Sigma^2 ku$, and this turns out to be a lift to ku-theory of Sq^2, which, together with $E(1)$, generates $\mathcal{A}(1)$.

This takes on additional force in connection with the Adams spectral sequence

$$\mathrm{Ext}_{\mathcal{A}}(H^*Y, H^*X) \Longrightarrow [X, Y]_2^\wedge$$

converging to the 2-completion of the module of homotopy classes of maps from X to Y. With $E = H\mathbf{Z}$, ku, or ko and $B = E(0)$, $E(1)$, or $\mathcal{A}(1)$ respectively, we get

$$\mathrm{Ext}_{\mathcal{A}}(H^*E, H^*X) \Longrightarrow [X, E]_2^\wedge = (E^*X)_2^\wedge$$

and

$$\mathrm{Ext}_{\mathcal{A}}(H^*(E \wedge X), \mathbf{F}_2) \Longrightarrow [S, E \wedge X]_2^\wedge = (E_*X)_2^\wedge$$

By the isomorphisms above, these can be written

$$\mathrm{Ext}_{\mathcal{A}}(\mathcal{A} \otimes_B \mathbf{F}_2, H^*X) \Longrightarrow [X, E]_2^\wedge = (E^*X)_2^\wedge$$

and

$$\mathrm{Ext}_{\mathcal{A}}(\mathcal{A} \otimes_B H^*X), \mathbf{F}_2) \Longrightarrow [S, E \wedge X]_2^\wedge = (E_*X)_2^\wedge$$

and by standard change of rings isomorphisms, this is

$$\mathrm{Ext}_B(\mathbf{F}_2, H^*X) \Longrightarrow [X, E]_2^\wedge = (E^*X)_2^\wedge$$

and

$$\mathrm{Ext}_B(H^*X), \mathbf{F}_2) \Longrightarrow [S, E \wedge X]_2^\wedge = (E_*X)_2^\wedge$$

In particular, the structure of H^*X as a module over the semi-dihedral algebra $\mathcal{A}(1)$ determines the E_2-term of the Adams spectral sequences converging to ko^*X and to ko_*X.

As usual, E_2 is not the end of the story, and it turns out that differentials in these spectral sequences are affected by the rest of the $\mathcal{A}$-module structure of H^*X. This should be understood as telling us about the relation between the rest of the Steenrod algebra and $\mathrm{End}(E)$.

I should admit that we have not used the structure theory of Crawley-Boevey. In fact, we are working over $\mathbf{F}_2$ so would have had to do the Galois descent from $\mathbf{F}_4$ to $\mathbf{F}_2$ to do so. But more significantly, the small number of modules we have seen in algebraic topology can be dealt with on an *ad hoc* basis as they occur.

Note also, that the relation between ko and ku, and its reflection in the relation between $E(1)$ and $\mathcal{A}(1)$, suggests that we should be able to classify modules over $\mathcal{A}(1)$ by some sort of Galois theory together with the extremely simple classification of modules over the exterior algebra on two generators, $E(1)$.

References

1. Atiyah, *K-Theory and Reality.*
2. Atiyah, Bott, and Shapiro.
3. R. R. Bruner, and J.P.C.Greenlees, *The Connective K-Theory of Finite Groups*, Memoirs AMS, 2004.
4. Crawley-Boevey, *Functorial Filtrations I, II, and III.*
5. *The Steenrod Algebra and its Dual.*

Mathematics Department, Wayne State University, Detroit, Michigan, 48202, USA
E-mail address: rrb@math.wayne.edu