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Some root invariants and Steenrod operations in $\mathrm{Ext}_A(F_2, F_2)$

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Abstract. We give the results of computations of root invariants in Ext over the Steenrod algebra through the 25-stem, with partial information through the 45-stem. This allows the computation of some new Steenrod operations as well.

These calculations were started to provide evidence for or against the algebraic Bredon-Löffler conjecture. This is the conjecture that

$$\eta_k^* : \mathrm{Ext}_A^{s,t}(F_2, F_2) \longrightarrow \mathrm{Ext}_A^{s,t}(\Sigma L_{-k}, F_2)$$

is a monomorphism for $0 < t - s < k/2$. Here L is the Laurent series ring $F_2[x, x^{-1}]$ with its usual Steenrod algebra action, L_{-k} is the submodule of L consisting of those elements whose degree is greater than or equal to $-k$, and $\eta_k : \Sigma L_{-k} \longrightarrow F_2$ is the nonzero homomorphism. The calculations of η_k^* showed [**5**] that the conjecture holds for $k \leq 55$. With those calculations in hand, we needed only calculate the homomorphisms

$$i_k^* : \mathrm{Ext}_A^{s,t}(\Sigma^{1-k} F_2, F_2) \longrightarrow \mathrm{Ext}_A^{s,t}(\Sigma L_{-k}, F_2)$$

induced by the projections onto the bottom dimensions, $i_k : \Sigma L_{-k} \longrightarrow \Sigma^{1-k} F_2$, to calculate all the root invariants through the 25 stem, and about half of those in the 26 through 45 stems. This extends the calculations of Mahowald and Shick [**7**], who found all the root invariants through the 16 stem. Our calculations were done by first using the programs described in [**1, 2**] to compute resolutions of the modules L_{-k} and the chain maps induced by η_k and i_k. We then used MAGMA to compute the induced maps of Ext and the root invariants. A calculation of $\mathrm{Ext}_A(F_2, F_2)$ for $s < 40$ and $t \leq 140$ with complete information on the product structure was calculated in the same manner: using MAGMA to process the

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output of the minimal resolution programs [**1, 2**]. A preliminary version of this can be found in [**3**], where the notation for elements of $\mathrm{Ext}_A(F_2, F_2)$ used here is defined. Charts showing the results of the Ext calculations (but not the induced homomorphisms) can be found at http://www.math.wayne.edu/~rrb/cohom.

By [**6**, Proposition 2.5], we may conclude that $Sq^0(x) \in R(x)$ when $|Sq^0(x)| = |R(x)|$, that is, if $k = n + s + 1$. This allows us to use the calculations here to determine a number of Steenrod operations.

THEOREM 1. *In the cohomology of the mod 2 Steenrod algebra*
(i) $Sq^0(Ph_1) = h_2 g$
(ii) $Sq^0(Pc_0) = h_1 y$
(iii) $Sq^0(Pd_0) = gd_1$
(iv) $Sq^0(i) = h_2 C$
(v) $Sq^0(k) = h_2 h_5 n + \epsilon h_3 D_2$, *where* $\epsilon \in \{0, 1\}$
(vi) $Sq^0(q) = h_2 Q_3$
(vii) $Sq^0(l) = h_0 x_{6,47}$
(viii) $Sq^0(y) = h_4 Q_3 + h_2 x_1$

These are the Sq^0's which are not 'tautologous'. In addition, we have 12 more which allow us to identify, in the resolution produced by the machine, the elements named. That is, we know by definition that $e_1 = Sq^0(e_0)$, for example. However, we have no way *a priori* to determine which of two elements in the 38 stem is e_1 since we have no general mechanical method of computing Steenrod operations. Thus, the root invariant is quite useful, in that it unambiguously identifies e_1 for us. The elements identified in this manner are n_1, p_1, e_1, e_2, f_1, and f_2. We also determine m_1 and t_1 up to F_2 indeterminacy.

Finally, note that the root invariant $R(h_2^2 g) = r_1$ shows that the Strong Algebraic Bredon-Löffler Conjecture in [**4**] must have its $(t-s)$-intercept decreased by at least 3. This is an inessential modification, as the key observation in these calculations supporting the conjecture is the slope -1/2.

CONJECTURE 2. *(Strong Algebraic Bredon-Löffler Conjecture) The map* η_k^* *is a monomorphism if*

$$s < (k - n - 3)/2.$$

KEY TO THE TABLES: Each row in the table lists the filtration s and stem n of an element $x \in \mathrm{Ext}_A(F_2, F_2)$, its root invariant $R(x)$, the dimension k of the cell on which it occurs, whether $Sq^0(x) \in R(x)$, and a basis for the indeterminacy in $R(x)$. The root invariant $R(x)$ is in the $n + k - 1$ stem. By [**6**, Proposition 2.5], we may test whether or not $Sq^0(x) \in R(x)$ by checking whether or not $|Sq^0(x)| = |R(x)|$, that is, $k = n + s + 1$.

k	s	n	x	$R(x)$	$Sq^0 \in R(x)$?	Indeterminacy
2	1	0	h_0	h_1	yes	
3	2	0	h_0^2	h_1^2	yes	
4	3	0	h_0^3	$h_0^2h_2$	yes	
8	4	0	h_0^4	$h_0^3h_3$	no	
10	5	0	h_0^5	Ph_1	no	
11	6	0	h_0^6	h_1Ph_1	no	
12	7	0	h_0^7	$h_0^2Ph_2$	no	
16	8	0	h_0^8	$h_0^7h_4$	no	
18	9	0	h_0^9	P^2h_1	no	
19	10	0	h_0^{10}	$h_1P^2h_1$	no	
20	11	0	h_0^{11}	$h_0^2P^2h_2$	no	
24	12	0	h_0^{12}	h_0^5i	no	
26	13	0	h_0^{13}	P^3h_1	no	
27	14	0	h_0^{14}	$h_1P^3h_1$	no	
28	15	0	h_0^{15}	$h_0^2P^3h_2$	no	
32	16	0	h_0^{16}	$h_0^{15}h_5$	no	
34	17	0	h_0^{17}	P^4h_1	no	
35	18	0	h_0^{18}	$h_1P^4h_1$	no	
36	19	0	h_0^{19}	$h_0^2P^4h_2$	no	
3	1	1	h_1	h_2	yes	
5	2	2	h_1^2	h_2^2	yes	
5	1	3	h_2	h_3	yes	
6	2	3	h_0h_2	h_1h_3	yes	
7	3	3	$h_0^2h_2$	$h_1^2h_3$	yes	
9	2	6	h_2^2	h_3^2	yes	
9	1	7	h_3	h_4	yes	
10	2	7	h_0h_3	h_1h_4	yes	
11	3	7	$h_0^2h_3$	$h_1^2h_4$	yes	
12	4	7	$h_0^3h_3$	$h_0^2h_2h_4$	yes	
11	2	8	h_1h_3	h_2h_4	yes	
12	3	8	c_0	c_1	yes	
13	3	9	$h_1^2h_3$	$h_2^2h_4$	yes	
14	4	9	h_1c_0	h_2c_1	yes	
15	5	9	Ph_1	h_2g	yes	
21	6	10	h_1Ph_1	r	no	
20	5	11	Ph_2	$h_0^3h_4^2$	no	
22	6	11	h_0Ph_2	q	no	
23	7	11	$h_0^2Ph_2$	h_1q	no	
17	2	14	h_3^2	h_4^2	yes	
18	3	14	$h_0h_3^2$	$h_1h_4^2$	yes	
19	4	14	d_0	d_1	yes	

k	s	n	x	$R(x)$	$Sq^0 \in R(x)$?	Indeterminacy
20	5	14	h_0d_0	h_0p	yes	
25	6	14	$h_0^2d_0$	y	no	h_1x
17	1	15	h_4	h_5	yes	
18	2	15	h_0h_4	h_1h_5	yes	
19	3	15	$h_0^2h_4$	$h_1^2h_5$	yes	
20	4	15	$h_0^3h_4$	$h_0^2h_2h_5$	yes	
21	5	15	h_1d_0	h_2d_1	yes	
24	5	15	$h_0^4h_4$	$h_0^3h_3h_5$	no	
26	6	15	$h_0^5h_4$	h_5Ph_1	no	$h_0^2f_1$
27	7	15	$h_0^6h_4$	$h_1h_5Ph_1$	no	
28	8	15	$h_0^7h_4$	$h_0^2h_5Ph_2$	no	
19	2	16	h_1h_4	h_2h_5	yes	
23	6	16	$h_1^2d_0$	h_1x	yes	
24	7	16	Pc_0	h_1y	yes	
21	3	17	$h_1^2h_4$	$h_2^2h_5$	yes	
22	4	17	e_0	e_1	yes	
23	5	17	h_0e_0	h_1e_1	yes	
24	6	17	$h_0^2e_0$	$h_0^2f_1$	yes	
31	7	17	$h_0^3e_0$	$h_1^2h_5d_0$	no	
30	8	17	h_1Pc_0	N	no	
32	9	17	P^2h_1	$h_1h_5Pc_0$	no	
21	2	18	h_2h_4	h_3h_5	yes	
22	3	18	$h_0h_2h_4$	$h_1h_3h_5$	yes	
23	4	18	f_0	f_1	yes	
23	4	18	$h_0^2h_2h_4$	$h_1^2h_3h_5$	yes	
24	5	18	h_0f_0	$h_0^2c_2$	yes	
37	10	18	$h_1P^2h_1$	R_1	no	$h_0^2h_5i$
23	3	19	c_1	c_2	yes	
35	9	19	P^2h_2	h_5Pd_0	no	
36	10	19	$h_0P^2h_2$	$h_0^2h_5i$	no	
39	11	19	$h_0^2P^2h_2$	h_1Q_1	no	
25	4	20	g	g_2	yes	
26	5	20	h_0g	h_1g_2	yes	
28	6	20	h_0^2g	$h_0h_2g_2$	no	
25	3	21	$h_2^2h_4$	$h_3^2h_5$	yes	
27	5	21	h_1g	h_2g_2	yes	
27	4	22	h_2c_1	h_3c_2	yes	
31	8	22	Pd_0	gd_1	yes	
36	9	22	h_0Pd_0	$h_0^2Q_2$	no	
41	10	22	$h_0^2Pd_0$	$x_{10,27}$	no	$x_{10,28}, h_1X_1$
28	4	23	h_4c_0	h_5c_1	yes	

k	s	n	x	$R(x)$	$Sq^0 \in R(x)$?	Indeterminacy
29	5	23	h_2g	h_3g_2	yes	
30	6	23	h_0h_2g	$h_1h_3g_2$	yes	
31	7	23	i	h_2C	yes	
36	8	23	h_0i	$h_0^2D_2$	no	
38	9	23	h_0^2i	$h_0^2B_3$	no	
40	9	23	$h_0^2i + h_1Pd_0$	$h_0^7h_5^2$	no	
42	10	23	h_0^3i	$x_{10,32}$	no	$h_0^2h_3Q_2$
43	11	23	h_0^4i	$h_1x_{10,32}$	no	h_0B_{23}
44	12	23	h_0^5i	$h_0^2PD_2$	no	
30	5	24	$h_1h_4c_0$	$h_2h_5c_1$	yes	
43	10	24	$h_1^2Pd_0$	$B_5 + PD_2$	no	
44	11	24	P^2c_0	$x_{11,35} + h_0^2x_{9,40}$	no	
38	8	25	Pe_0	$x_{8,32} + x_{8,33}$	no	
39	9	25	h_0Pe_0	$h_1x_{8,32}$	no	$h_0^2x_{7,33}, h_0^8h_6$
40	10	25	$h_0^2Pe_0$	$h_0^2h_3Q_2$	no	
45	11	25	$h_0^3Pe_0$	h_2B_5	no	h_0PA
46	12	25	$h_1P^2c_0$	$h_2x_{11,35}$	no	
47	13	25	P^3h_1	$x_{13,34}$	no	$x_{13,35}$
41	6	26	h_2^2g	r_1	no	
36	7	26	j	h_0A'	no	
39	8	26	h_0j	h_3Q_2	no	h_0h_2A, h_0^2A''
40	9	26	h_0^2j	$h_0^2h_3D_2$	no	
51	13	27	P^3h_2	e_0B_4	no	$h_0^6x_{7,57}$
41	8	28	d_0^2	G_{21}	no	h_0h_3A'
42	9	28	$h_0d_0^2$	h_1G_{21}	no	$h_0h_2x_{7,40}$
44	10	28	$h_0^2d_0^2$	$h_0h_2G_{21}$	no	
37	7	29	k	h_2h_5n	yes	h_3D_2
40	8	29	h_0k	h_0h_3A'	no	
43	9	29	h_0^2k	h_2G_{21}	no	
33	2	30	h_4^2	h_5^2	yes	
34	3	30	$h_0h_4^2$	$h_1h_5^2$	yes	
35	4	30	$h_0^2h_4^2$	$h_1^2h_5^2$	yes	
36	5	30	$h_0^3h_4^2$	$h_0^2h_2h_5^2$	yes	
37	6	30	r	r_1	yes	
39	7	30	h_0r	$h_2^2H_1 + h_3A'$	no	
41	8	30	h_0^2r	$h_3x_{7,33}$	no	
44	9	30	h_0^3r	$h_0^2h_4D_2$	no	
49	10	30	h_0^4r	$P^2h_5^2$	no	e_0A'
50	11	30	h_0^5r	$h_1P^2h_5^2$	no	
51	12	30	P^2d_0	$x_{12,44}$	no	
33	1	31	h_5	h_6	yes	

k	s	n	x	$R(x)$	$Sq^0 \in R(x)$?	Indeterminacy
34	2	31	h_0h_5	h_1h_6	yes	
35	3	31	$h_0^2h_5$	$h_1^2h_6$	yes	
35	3	31	$h_1h_4^2$	$h_2h_5^2$	yes	
36	4	31	$h_0^3h_5$	$h_0^2h_2h_6$	yes	
37	5	31	n	n_1	yes	
40	5	31	$h_0^4h_5$	$h_0^3h_3h_6$	no	
42	6	31	$h_0^5h_5$	h_6Ph_1	no	
43	7	31	$h_0^6h_5$	$h_1h_6Ph_1$	no	$h_2^2Q_3$
40	8	31	d_0e_0	$h_3x_{7,33}$	yes	
44	8	31	$h_0^7h_5$	$h_0^2h_6Ph_2$	no	
45	9	31	$h_0d_0e_0$	h_3G_{21}	no	$h_0^2x_{7,53}$
48	9	31	$h_0^8h_5$	$h_0^7h_4h_6$	no	
46	10	31	$h_0^2d_0e_0$	$h_0x_{9,51}$	no	
50	10	31	$h_0^9h_5$	$h_6P^2h_1$	no	
51	11	31	$h_0^{10}h_5$	$h_1h_6P^2h_1$	no	h_0gA'
35	2	32	h_1h_5	h_2h_6	yes	
37	4	32	d_1	d_2	yes	
39	6	32	q	h_2Q_3	yes	
40	7	32	l	$h_0x_{6,47}$	yes	
43	8	32	h_0l	$x_{8,51}$	no	$h_0^2h_6Ph_2$
44	9	32	h_0^2l	$h_0^2x_{7,53}$	no	
37	3	33	$h_1^2h_5$	$h_2^2h_6$	yes	
38	4	33	p	p_1	yes	
39	5	33	h_0p	h_1p_1	yes	
45	7	33	h_1q	$x_{7,57}$	no	$h_1x_{6,53}, h_0^2h_6d_0$
37	2	34	h_2h_5	h_3h_6	yes	
38	3	34	$h_0h_2h_5$	$h_1h_3h_6$	yes	
39	4	34	$h_0^2h_2h_5$	$h_1^2h_3h_6$	yes	
41	6	34	h_2n	h_3n_1	yes	h_6Ph_2
43	8	34	d_0g	$h_1x_{7,53}$	yes	$h_0^2x_{6,53}$
44	9	34	h_0d_0g	$h_0^2m_1$	yes	
48	11	34	Pj	h_0gA'	no	
41	5	35	h_2d_1	h_3d_2	yes	
43	7	35	m	m_1	yes	$h_0^2h_6d_0$
44	8	35	h_0m	$h_0^2t_1$	yes	
47	9	35	h_0^2m	$h_3x_{8,51}$	no	
43	6	36	t	t_1	yes	$h_0^4h_4h_6$
41	3	37	$h_2^2h_5$	$h_3^2h_6$	yes	
43	5	37	x	x_1	yes	
44	6	37	h_0x	$h_0^2e_2$	yes	
45	7	37	h_1t	h_2t_1	yes	

k	s	n	x	$R(x)$	$Sq^0 \in R(x)$?	Indeterminacy
46	7	37	$h_0^2 x$	$h_0 h_2 x_1$	no	
46	8	37	$e_0 g$	$e_1 g_2$	yes	
48	8	37	$h_0^3 x$	$h_0^4 f_2$	no	
41	2	38	$h_3 h_5$	$h_4 h_6$	yes	
42	3	38	$h_0 h_3 h_5$	$h_1 h_4 h_6$	yes	
43	4	38	$h_0^2 h_3 h_5$	$h_1^2 h_4 h_6$	yes	
43	4	38	e_1	e_2	yes	
44	5	38	$h_0^3 h_3 h_5$	$h_0^2 h_2 h_4 h_6$	yes	
45	6	38	$h_1 x$	$h_2 x_1$	yes	
45	6	38	y	$h_4 Q_3 + h_2 x_1$	yes	
46	7	38	$h_0 y$	$h_1 h_4 Q_3$	yes	$h_0^2 h_6 g$
48	8	38	$h_0^2 y$	$h_0^5 c_3$	no	
43	3	39	$h_1 h_3 h_5$	$h_2 h_4 h_6$	yes	
44	4	39	$h_5 c_0$	$h_6 c_1$	yes	
45	5	39	$h_1 e_1$	$h_2 e_2$	yes	
49	7	39	$h_1 y$	$x_{7,74}$	no	
45	4	40	$h_1^2 h_3 h_5$	$h_2^2 h_4 h_6$	yes	
45	4	40	f_1	f_2	yes	
46	5	40	$h_0 f_1$	$h_1 f_2$	yes	
46	5	40	$h_1 h_5 c_0$	$h_2 h_6 c_1$	yes	
47	6	40	$h_0^2 f_1$	$h_1^2 f_2$	yes	
47	6	40	$h_5 P h_1$	$h_2 h_6 g$	yes	
45	3	41	c_2	c_3	yes	
46	4	41	$h_0 c_2$	$h_1 c_3$	yes	
47	5	41	$h_0^2 c_2$	$h_1^2 c_3$	yes	
49	4	44	g_2	g_3	yes	
49	3	45	$h_3^2 h_5$	$h_4^2 h_6$	yes	

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