OSSA'S THEOREM AND ADAMS COVERS

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Abstract. We show that Ossa's theorem splitting $ku \wedge BV$ for elementary abelian groups V follows from general facts about $ku \wedge BZ/2$ and Adams covers. For completeness, we also provide the analogous results for $ko \wedge BV$.

1. Introduction

The connective complex K-homology of BV, V elementary abelian, has been computed by Ossa [4], and, by different techniques, by Johnson and Wilson [1]. A key step in both proofs is the equivalence of ku-module spectra

$$ku \wedge BZ/2 \wedge BZ/2 \ \simeq \ ku \wedge \Sigma^2 BZ/2 \ \vee \ \Sigma^2 H(\mathbb{F}_2[u,v])$$

where $H(\mathbb{F}_2[u,v])$ is the generalized Eilenberg-MacLane spectrum on the graded vector space $\mathbb{F}_2[u,v]$ with $|u| = |v| = 2$.

The purpose of this note is to show that this equivalence follows from two more general facts of independent interest about such smash products. We also provide analogous results in the real case, and the splitting

$$ko \wedge BZ/2 \wedge BZ/2 \ \simeq \ ko{<}1{>} \wedge BZ/2 \ \vee \ \Sigma^2 H(\mathbb{F}_2[u,v^2]),$$

where $ko{<}1{>}$ is the connected cover of ko, that is, the fiber of the Thom map $ko \longrightarrow H\mathbb{Z}$.

The calculation of $ku \wedge BV$ and $ko \wedge BV$ from these splittings follows from formal properties of products, and the reader is referred to [4] or [1] for details. Results analogous to the complex case hold for both the real and complex cases at odd primes.

I would like to thank Stephan Stolz for introducing me to the essential cofiber sequence (1) as the simplest way to calculate $ku_*BZ/2$ and $ko_*BZ/2$. Its use for this purpose is probably due to Mark Mahowald or Don Davis.

2. Generalities

As $BZ/2$ is a 2-local spectrum, we shall localize at 2 throughout.

Notation 2.1. *Let $H = H\mathbb{F}_2$ and let 'cohomology' mean mod 2 cohomology. Let $\mathcal{A}$ be the mod 2 Steenrod algebra, $\mathcal{A}(i)$ the subalgera generated by $Sq^1, \ldots, Sq^{2^i}$, and $E(1) = E[Q_0, Q_1]$ the exterior subalgebra of $\mathcal{A}(1)$ generated by $Q_0 = Sq^1$ and $Q_1 = [Sq^2, Q_0]$.*

Let $P = BZ/2$ and let R be the suspension of the quotient of the stunted projective space P_{-1}^{∞} by the 0-cell. (Thus, $H^*R = \Sigma F_{-1,1}$ in the notation of [2].)

The bottom cell of R is in dimension 0, and the fiber of the inclusion of the bottom cell is P. This gives the key cofiber sequence

$$P \longrightarrow S \longrightarrow R. \qquad (1)$$

Lemma 2.2.

$$ko \wedge R \;\simeq\; \bigvee_{\mathrm{i}\geq 0} \Sigma^{4i} H\mathbb{Z}$$

and

$$ku \wedge R \;\simeq\; \bigvee_{\mathrm{i}\geq 0} \Sigma^{2i} H\mathbb{Z}$$

Proof: It suffices to prove the first equivalence, since $ku \;\simeq\; ko \wedge C\eta$. In [2] it is shown that

$$\mathcal{A} \underset{\mathcal{A}(1)}{\otimes} H^*R \;\cong\; \bigoplus_{\mathrm{i}\geq 0} \mathcal{A} \underset{\mathcal{A}(0)}{\otimes} \Sigma^{4i}\mathbb{F}_2. \tag{2}$$

Since $H^*(ko \wedge R) \;\cong\; (\mathcal{A} \underset{\mathrm{A}(1)}{\otimes} \mathbb{F}_2) \otimes H^*R \;\cong\; \mathcal{A} \underset{\mathrm{A}(1)}{\otimes} H^*R$, the isomorphism (2) implies that the Adams spectral sequence

$$\mathrm{Ext}_{\mathcal{A}}(H^*(ko \wedge R), \mathbb{F}_2) \Longrightarrow \pi_*(ko \wedge R)$$

is isomorphic, by a standard change of rings argument, to

$$\bigoplus_{\mathrm{i}\geq 0} \mathrm{Ext}_{\mathcal{A}(0)}(\Sigma^{4i}\mathbb{F}_2, \mathbb{F}_2) \Longrightarrow \pi_*(ko \wedge R).$$

This must collapse at E_2 because it is zero except in total degrees congruent to 0 modulo 4. If we let $x_i : ko \wedge R \longrightarrow \Sigma^{4i} H\mathbb{Z}$ be a cohomology class dual to the generator of $\mathrm{Ext}^{0,4i}(H^*(ko \wedge R), \mathbb{F}_2)$ then the collapse of the Adams spectral sequence implies that

$$\bigvee_{\mathrm{i}\geq 0} x_i : ko \wedge R \longrightarrow \bigvee_{\mathrm{i}\geq 0} \Sigma^{4i} H\mathbb{Z}$$

induces an isomorphism in homotopy, and is thus a homotopy equivalence. □

Theorem 2.3. *For any spectrum X there are cofiber sequences*

$$ko \wedge P \wedge X \xrightarrow{i} ko \wedge X \xrightarrow{p} \bigvee_{\mathrm{i}\geq 0} \Sigma^{4i} H\mathbb{Z} \wedge X$$

and

$$ku \wedge P \wedge X \xrightarrow{i} ku \wedge X \xrightarrow{p} \bigvee_{\mathrm{i}\geq 0} \Sigma^{2i} H\mathbb{Z} \wedge X$$

in which the homomorphisms p^* are epimorphisms in cohomology.

Proof: Smash the cofiber sequence (1) with $ko \wedge X$ or $ku \wedge X$ and apply Lemma 2.2. Since tensor products preserve epimorphisms, by the Kunnëth theorem it is sufficient to note that $H^*R \longrightarrow H^*S$ is an epimorphism. □

Thus, smashing $ko \wedge X$ or $ku \wedge X$ with P has the effect of taking a 'generalized Adams cover'. This is not in general an actual $H\mathbb{Z}$ Adams cover because $H\mathbb{Z} \wedge X$ is not in general a coproduct of $H\mathbb{Z}$'s. Further, this cover contains a large generalized Eilenberg-MacLane summand, so is far from minimal.

3. The Complex Case

In the complex case we have a much smaller generalized Adams cover.

Proposition 3.1. *For any spectrum X there is a cofiber sequence*

$$ku \wedge \Sigma^2 X \xrightarrow{u} ku \wedge X \xrightarrow{p} H\mathbb{Z} \wedge X,$$

where $u \in ku_2$ is the multiplicative generator of ku_* and p^* is an epimorphism in cohomology.

Proof: Multiplication by the Bott class $u \in ku_2$ gives a cofiber sequence

$$\Sigma^2 ku \xrightarrow{u} ku \longrightarrow H\mathbb{Z}$$

in which the map $ku \longrightarrow H\mathbb{Z}$ induces on H^* the quotient

$$\mathcal{A} \underset{\mathrm{E}(1)}{\otimes} \mathbb{F}_2 \longleftarrow \mathcal{A} \underset{\mathcal{A}(0)}{\otimes} \mathbb{F}_2.$$

This remains an epimorphism upon tensoring with H^*X. $\square$

We would like to claim that this smaller generalized Adams cover of $ku \wedge X$ splits off the one in Theorem 2.3. However, there is no comparison theorem for $H\mathbb{Z}$ Adams resolutions. For example, the map $ku \longrightarrow ku \wedge R$ of Theorem 2.3 does not factor through the map $ku \longrightarrow H\mathbb{Z}$ of Proposition 3.1. The standard theorems which assert the existence of a comparison theorem would require that $H\mathbb{Z}^*(H\mathbb{Z})$ be projective over $\pi_* H\mathbb{Z}$, which it is not. However, with coefficients in a field, such a comparison theorem does exist and allows us to compare the cofibrations above.

Theorem 3.2. *If H^*X is free over $E[\beta]$ and $ku_*X \longrightarrow H\mathbb{Z}_*X$ is onto, then*

$$ku \wedge P \wedge X \;\simeq\; ku \wedge \Sigma^2 X \;\vee\; \bigvee_{\mathrm{i}>0} \Sigma^{2i-1} H\mathbb{Z} \wedge X.$$

Proof: If H^*X is $E[\beta]$-free then $H\mathbb{Z}^*X$ is a mod 2 vector space, so $H\mathbb{Z} \wedge X$ is a wedge of $H = H\mathbb{F}_2$'s. Hence, the sequences in Theorems 2.3 and 3.1 are both beginnings of $H\mathbb{F}_2$-Adams resolutions. As such, the comparison theorem gives maps between them which cover the identity of $ku \wedge X$.

$$\begin{array}{ccccc}
ku \wedge \Sigma^2 X & \longrightarrow & ku \wedge P \wedge X & \longrightarrow & ku \wedge \Sigma^2 X \\
\downarrow & & \downarrow & & \downarrow \\
ku \wedge X & = & ku \wedge X & = & ku \wedge X \\
\downarrow & & \downarrow & & \downarrow \\
H\mathbb{Z} \wedge X & \longrightarrow & \bigvee_{\mathrm{i}\geq 0} \Sigma^{2i} H\mathbb{Z} \wedge X & \longrightarrow & H\mathbb{Z} \wedge X
\end{array}$$

Since $ku_*X \longrightarrow H\mathbb{Z}_*X$ is onto, the bottom row must be the identity in homotopy, hence an equivalence. It follows that we have a splitting

$$ku \wedge P \wedge X \;\simeq\; ku \wedge \Sigma^2 X \;\vee\; \bigvee_{\mathrm{i}>0} \Sigma^{2i-1} H\mathbb{Z} \wedge X$$

since the left map in the bottom row may be taken to be the inclusion of the $i = 0$ summand.

$\square$

Corollary 3.3.

$$ku \wedge P \wedge P \;\simeq\; ku \wedge \Sigma^2 P \;\vee\; \bigvee_{\mathrm{i,j}>0} \Sigma^{2i+2j-2} H\mathbb{F}_2.$$

Proof: This follows from Theorem 3.2 and the equivalence

$$H\mathbb{Z} \wedge P \;\simeq\; \bigvee_{\mathrm{j}>0} \Sigma^{2j-1} H\mathbb{F}_2$$

since it is easy to see that $ku_*P \longrightarrow H\mathbb{Z}_*P$ is onto. □

Remark 3.4. *We get the dimensions right if we write the generalized Eilenberg-MacLane summand as the double suspension of $H(\mathbb{F}_2[u,v])$. However, the geometric origin of the splitting is clearer with the following parameterization. We have $H^*(P \wedge P) = (x_1x_2)$, the ideal generated by x_1x_2 in $H^*(P \times P)_+ = \mathbb{F}_2[x_1, x_2]$. As an $E(1)$-module, each $x_1^{2i-1}x_2^{2j-1}$ generates a copy of $E(1)$ and these form an $E(1)$-free submodule*

$$\bigoplus_{\mathrm{i,j}>0} <x_1^{2i-1}x_2^{2j-1}> \;\subset\; H^*(P \wedge P).$$

Since $E(1)$ is a Frobenius algebra, this submodule is a direct summand and it is easy to check that the quotient is isomorphic to $H^*(\Sigma^2 P)$. For example, we can write

$$H^*(P \wedge P) \;\cong\; x_1^2(x_2) \oplus \bigoplus_{\mathrm{i,j}>0} <x_1^{2i-1}x_2^{2j-1}>$$

Since Q_0 and Q_1 act trivially on x_1^2, the first summand is $H^*(\Sigma^2 P)$ as an $E(1)$-module. If all one wants is to understand $ku \wedge P \wedge P$, this splitting of cohomology and Margolis's theorem, that $\mathcal{A}$-free submodules in cohomology correspond to $H\mathbb{F}_2$ wedge summands ([3]), are sufficient.

Remark 3.5. *In general, an Adams cover of $ku \wedge X$ is not just $ku \wedge \Sigma^2 X$. For example $X = BZ/4$ is composed of mod 4 Moore spaces, and $ku_*BZ/4 \longrightarrow H\mathbb{Z}_*BZ/4$ is onto, presumably the mildest weakening of the hypotheses of Theorem 3.2. We have the following homotopy groups for $ku \wedge BZ/4$ and its first three Adams covers, where we write $a+b+c$ for the group $\mathbb{Z}/a \oplus \mathbb{Z}/b \oplus \mathbb{Z}/c$ and 0 for the trivial group.*

Adams cover	π_1	π_3	π_5	π_7	π_9	π_{11}
0	*4*	*2+8*	*2+2+16*	*4+2+32*	*4+4+64*	*8+4+128*
1	*2*	*2+4*	*2+2+8*	*4+2+16*	*4+4+32*	*8+4+64*
2	*0*	*2*	*2+4*	*2+2+8*	*4+2+16*	*4+4+32*
3	*0*	*0*	*2*	*2+4*	*2+2+8*	*4+2+16*

From this it is clear that the first Adams cover is not merely a suspension, but it is plausible that the i^{th} Adams cover is the double suspension of the $(i-1)^{st}$ for $i > 1$.

4. The Real Case

We record the following trivial consequence of the definitions for reference.

Proposition 4.1. *For any spectrum X there is a cofiber sequence*

$$ko{<}1{>} \wedge X \longrightarrow ko \wedge X \;\xrightarrow{p}\; H\mathbb{Z} \wedge X,$$

where p^* is an epimorphism in cohomology.

As in the complex case, with coefficients in a field the comparison theorem allows us to compare the preceding cofibration with that in Theorem 2.3.

Theorem 4.2. *If H^*X is free over $E[\beta]$ and $ko_*X \longrightarrow H\mathbb{Z}_*X$ is onto then*

$$ko \wedge P \wedge X \;\simeq\; ko{<}1{>} \wedge X \;\vee\; \bigvee_{\mathrm{i}>0} \Sigma^{4i-1} H\mathbb{Z} \wedge X.$$

Proof: The proof is entirely analogous to the complex case. □

Remark 4.3. *It might be better to replace $ko<1> \wedge X$ by $(ko \wedge X)<1>$, a potentially smaller Adams cover of $ko \wedge X$. For example,*

$$ko{<}1{>} \wedge P \;\simeq\; (ko \wedge P){<}1{>} \;\vee\; \bigvee_{\mathrm{i}>0} \Sigma^{4i} H\mathbb{F}_2.$$

Theorem 4.4.

$$\begin{aligned} ko \wedge P \wedge P \quad &\simeq \quad ko{<}1{>} \wedge P \;\vee\; \bigvee_{\mathrm{i,j}>0} \Sigma^{4i+2j-2} H\mathbb{F}_2 \\ &\simeq \quad (ko \wedge P){<}1{>} \;\vee\; \bigvee_{\mathrm{i}>0,\mathrm{j}\geq 0} \Sigma^{4i+2j-2} H\mathbb{F}_2 \end{aligned}$$

where $(ko \wedge P){<}1{>}$ is the minimal Adams cover of $ko \wedge P$. The cohomology of $(ko \wedge P){<}1{>}$ is tensored up from an $\mathcal{A}(1)$-module with Poincare series $t^3/(1-t) + t^2(1+t)(1+t^3)$.

Proof: Theorem 4.2 does not apply, so we must resort to an ad hoc argument in this case. We decompose $H^*(P \wedge P)$ as a module over $\mathcal{A}(1)$ into an $\mathcal{A}(1)$-free summand, producing the generalized Eilenberg-MacLane spectrum upon smashing with ko, and the $\mathcal{A}(1)$-module which is the kernel of the minimal projective cover of H^*P in $\mathcal{A}(1)$-Mod. □

Remark 4.5. *The homotopy type of $ko \wedge P^{\wedge n}$ modulo generalized Eilenberg-MacLane summands depends upon the congruence class of n modulo 4. This is an easy consequence of the homotopy of the Adams covers of $ko \wedge P$, and is the maximal number of different homotopy types since the 4^{th} Adams cover of ko is $\Sigma^8 ko$.*

References

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