# ON THE POSTNIKOV TOWERS FOR REAL AND COMPLEX CONNECTIVE K-THEORY

ROBERT R. BRUNER

## 1. Introduction

The analysis of real connective K-theory is facilitated by the '$\eta c R$' cofiber sequence

$$\Sigma ko \xrightarrow{\eta} ko \xrightarrow{c} ku \xrightarrow{R} \Sigma^2 ko$$

relating real and complex K-theories [2]. Here we extend this relationship through the Postnikov towers, producing several useful $ko$-module maps in the process.

**Theorem 1.** *The $\eta c R$ sequence lifts to cofiber sequences relating the connective covers of $ko$ and $ku$ as follows:*

$$\begin{array}{ccccccc}
\Sigma ko & \xrightarrow{\eta} & ko & \xrightarrow{c} & ku & \xrightarrow{R} & \Sigma^2 ko \\
\Vert & & \uparrow & & \uparrow{\scriptstyle v} & & \Vert \\
\Sigma ko & \xrightarrow{\eta_1} & ko\langle 1\rangle & \xrightarrow{c_1} & \Sigma^2 ku & \xrightarrow{r} & \Sigma^2 ko \\
\uparrow & & \uparrow & & \uparrow{\scriptstyle v} & & \uparrow \\
\Sigma ko\langle 1\rangle & \xrightarrow{\eta_2} & ko\langle 2\rangle & \xrightarrow{c_2} & \Sigma^4 ku & \xrightarrow{r_1} & \Sigma^2 ko\langle 1\rangle \\
\uparrow & & \uparrow & & \Vert & & \uparrow \\
\Sigma ko\langle 2\rangle & \xrightarrow{\eta_4} & ko\langle 4\rangle & \xrightarrow{c_4} & \Sigma^4 ku & \xrightarrow{r_2} & \Sigma^2 ko\langle 2\rangle \\
\uparrow & & \uparrow & & \uparrow{\scriptstyle v} & & \uparrow \\
\Sigma ko\langle 4\rangle & \xrightarrow{\eta_8} & ko\langle 8\rangle & \xrightarrow{c_8} & \Sigma^6 ku & \xrightarrow{r_4} & \Sigma^2 ko\langle 4\rangle \\
\uparrow & & \Vert & & \uparrow{\scriptstyle v} & & \uparrow \\
\Sigma ko\langle 8\rangle & \xrightarrow{\Sigma^8\eta} & ko\langle 8\rangle & \xrightarrow{\Sigma^8 c} & \Sigma^8 ku & \xrightarrow{\Sigma^8 R} & \Sigma^2 ko\langle 8\rangle
\end{array}$$

In the sequence above, $c$ is complexification, $r$ is realification, and $\eta$ is multiplication by $\eta \in ko_1$. The map $R$ is an extension of realification $r$ over the Bott map: $r = Rv$.

We will write $X\langle n\rangle \longrightarrow X$ for the $n$-connected cover of $X$. By this we mean that $\pi_i X\langle n\rangle = 0$ for $i < n$, while $\pi_i X\langle n\rangle \longrightarrow \pi_i X$ is an isomorphism for $i \geq n$. It will be useful to record the maps induced in cohomology. All the modules and maps we will deal with are in the image of induction from $\mathcal{A}(1)$-Mod,

$$\mathcal{A} \otimes_{\mathcal{A}(1)} - : \mathcal{A}(1)\text{-Mod} \longrightarrow \mathcal{A}\text{-Mod},$$

so we will record the results in $\mathcal{A}(1)$-Mod, leaving it to the reader to tensor up.

The first lift, $\eta_1 c_1 r$, was brought to my attention by Vic Snaith ([3]). The remaining lifts appeared at one point to be useful in Geoffrey Powell's analysis of $ko^* BV_+$ ([4]), but in the end were unnecessary there.

2000 *Mathematics Subject Classification.* Primary: 55N15, 55S45; Secondary: 55N20, 55P42, 55R40, 55S05i, 55S10.

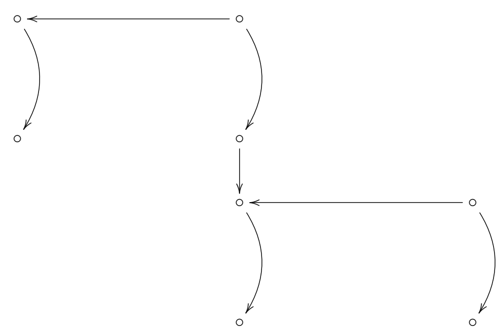
## 2. Complex Periodicity

In the complex case, periodicity and the Postnikov tower amount to the same thing. If we write $ku_* = \mathbf{Z}[v]$, with $|v| = 2$, then the Postnikov covers of $ku$ are simply given by multiplication by powers of $v$.

$$\begin{array}{ccc} \Sigma^2 ku & & \\ \downarrow\simeq & \searrow^{v} & \\ ku\langle 2\rangle & \longrightarrow & ku \end{array} \qquad \text{and more generally} \qquad \begin{array}{ccc} \Sigma^{2i+2} ku & \xrightarrow{v} & \Sigma^{2i} ku \\ \downarrow\simeq & & \downarrow\simeq \\ ku\langle 2i+2\rangle & \longrightarrow & ku\langle 2i\rangle \end{array}$$

**Proposition 2.** *$ku \longrightarrow H\mathbf{Z} \longrightarrow \Sigma^3 ku$ induces the short exact sequence*

$$\mathcal{A}(1)/(Sq^1, Sq^3) \longleftarrow \mathcal{A}(1)/(Sq^1) \longleftarrow \Sigma^3\mathcal{A}(1)/(Sq^1, Sq^3)$$



## 3. Real Periodicity

In the real case, periodicity is broken into 4 steps. We write $ko_* = \mathbf{Z}[\eta, \alpha, \beta]/(2\eta, \eta^3, \eta\alpha, \alpha^2 - 4\beta)$ with $|\eta| = 1$, $|\alpha| = 4$, and $|\beta| = 8$.

$$\begin{array}{ccccc} \Sigma^8 ko & \xrightarrow{\simeq} & ko\langle 8\rangle & & \\ & & \downarrow & & \\ & & ko\langle 4\rangle & \longrightarrow & \Sigma^4 H\mathbf{Z} \\ & & \downarrow & & \\ & & ko\langle 2\rangle & \longrightarrow & \Sigma^2 H\mathbf{F}_2 \\ & & \downarrow & & \\ & & ko\langle 1\rangle & \longrightarrow & \Sigma H\mathbf{F}_2 \\ & & \downarrow & & \\ & & ko & \longrightarrow & H\mathbf{Z} \end{array} \qquad \Sigma^8 ko \xrightarrow{\beta} ko$$

The following Proposition is well known. It is a simple way to show that a spectrum whose cohomology is $\mathcal{A}/\!\!/\mathcal{A}(1)$ must have 2-local homotopy additively isomorphic to $\pi_* ko$.

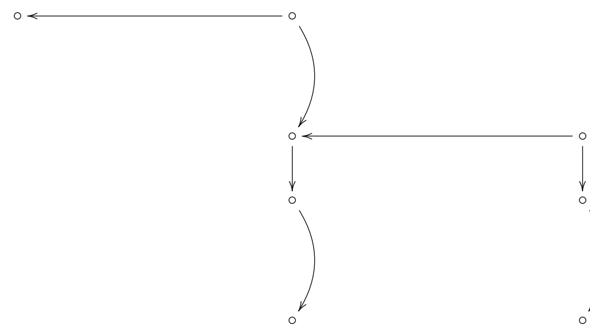
**Proposition 3.** *The maps induced in cohomology by the Postnikov tower for ko are as follows.*

(1)

$$ko \longrightarrow H\mathbf{Z} \longrightarrow \Sigma ko\langle 1\rangle$$

*induces the short exact sequence*

$$\mathbf{F}_2 \longleftarrow \mathcal{A}(1)/(Sq^1) \longleftarrow \Sigma^2\mathcal{A}(1)/(Sq^2)$$

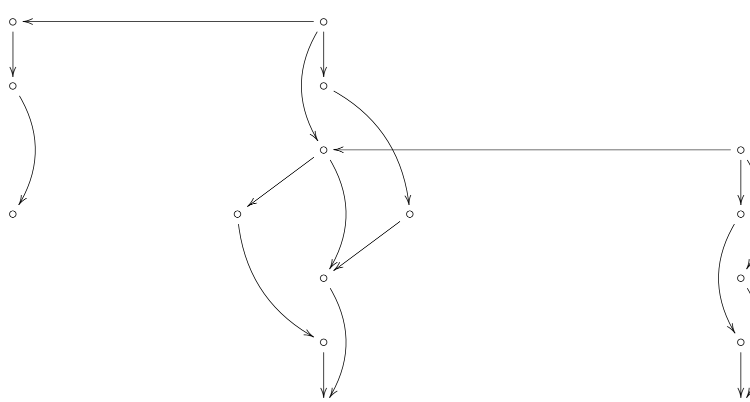


(2)

$$ko\langle 1\rangle \longrightarrow \Sigma H\mathbf{F}_2 \longrightarrow \Sigma ko\langle 2\rangle$$

*induces the short exact sequence*

$$\Sigma\mathcal{A}(1)/(Sq^2) \longleftarrow \Sigma\mathcal{A}(1) \longleftarrow \Sigma(Sq^2) \cong \Sigma^3\mathcal{A}(1)/(Sq^3)$$

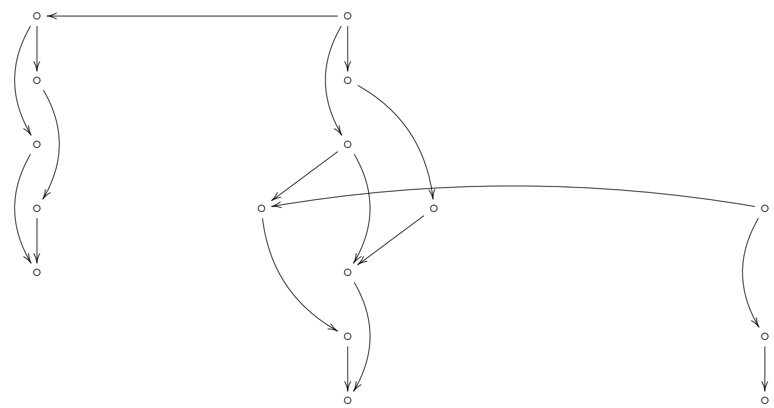


(3)

$$ko\langle 2\rangle \longrightarrow \Sigma^2 H\mathbf{F}_2 \longrightarrow \Sigma ko\langle 4\rangle$$

*induces the short exact sequence*

$$\Sigma^2\mathcal{A}(1)/(Sq^3) \longleftarrow \Sigma^2\mathcal{A}(1) \longleftarrow \Sigma^2(Sq^3) \;\cong\; \Sigma^5\mathcal{A}(1)/(Sq^1, Sq^2Sq^3)$$

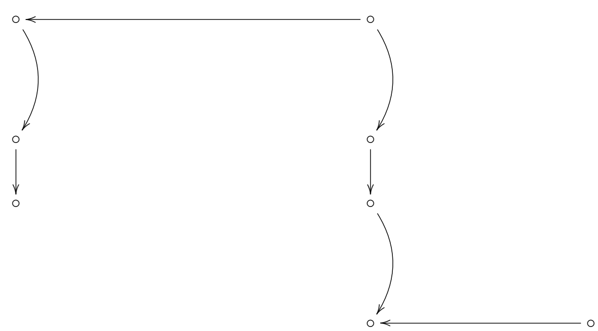


(4)

$$ko\langle 4\rangle \longrightarrow \Sigma^4 H\mathbf{Z} \longrightarrow \Sigma ko\langle 8\rangle$$

*induces the short exact sequence*

$$\Sigma^4\mathcal{A}(1)/(Sq^1, Sq^2Sq^3) \longleftarrow \mathcal{A}(1)/(Sq^1) \longleftarrow \Sigma^9\mathbf{F}_2$$

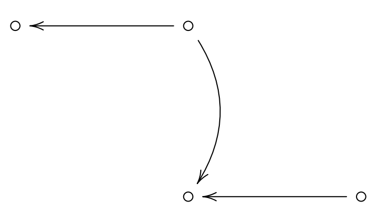


## 4. Maps of Postnikov towers

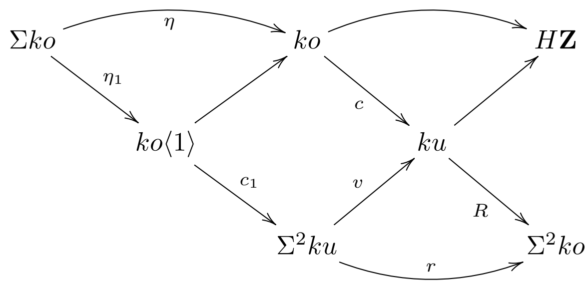
First, we record the maps induced in cohomology by our starting point, the $\eta cR$ sequence.

**Proposition 4.** $ko \xrightarrow{c} ku \xrightarrow{R} \Sigma^2 ko$ *induces the short exact sequence*

$$\mathcal{A}(1)/(Sq^1, Sq^2) \longleftarrow \mathcal{A}(1)/(Sq^1, Sq^3) \longleftarrow \Sigma^2\mathcal{A}(1)/(Sq^1, Sq^2)$$



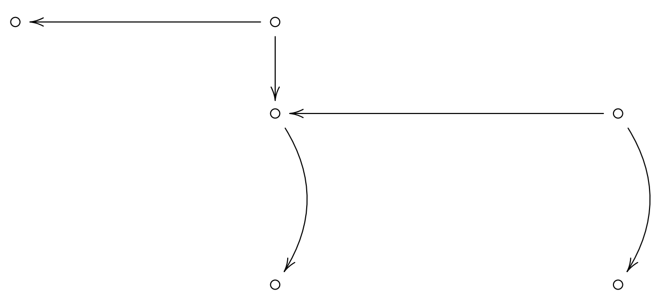
We will now prove Theorem 1 in a series of steps. We start with the braid of cofibrations induced by the composite $ko \xrightarrow{c} ku \longrightarrow H\mathbf{Z}$.




This gives the $\eta_1 c_1 r$ sequence. To continue to the next step, we will need to know the maps induced in cohomology by this one.

**Proposition 5.** *$\Sigma ko \xrightarrow{\eta_1} ko\langle 1\rangle \xrightarrow{c_1} \Sigma^2 ku$ induces the short exact sequence*

$$\Sigma\mathbf{F}_2 \longleftarrow \Sigma\mathcal{A}(1)/(Sq^2) \longleftarrow \Sigma^2\mathcal{A}(1)/(Sq^1, Sq^3)$$



*Proof.* These are the only maps which can make the long exact sequence exact. ∎

From this we observe that we have a commutative square

$$\begin{array}{ccc} \Sigma H\mathbf{F}_2 & \xrightarrow{Sq^1} & \Sigma^2 H\mathbf{Z} \\ \uparrow & & \uparrow \\ ko\langle 1\rangle & \xrightarrow{c_1} & \Sigma^2 ku \end{array}$$

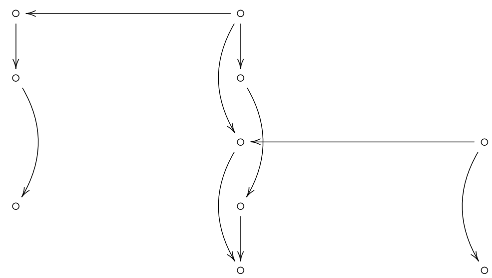
which induces the following map of cofiber sequences. The map induced in cohomology by $\eta_1$ implies that the left hand map $\Sigma ko \longrightarrow \Sigma H\mathbf{Z}$ is nontrivial. This implies that the fiber of $c_2$ is $\Sigma ko\langle 1\rangle$, giving the next Postnikov lift of the $\eta c R$ sequence.

$$\begin{array}{ccccccc} \Sigma H\mathbf{Z} & \longrightarrow & \Sigma H\mathbf{F}_2 & \xrightarrow{Sq^1} & \Sigma^2 H\mathbf{Z} & \longrightarrow & \Sigma^2 H\mathbf{Z} \\ \uparrow & & \uparrow & & \uparrow & & \uparrow \\ \Sigma ko & \xrightarrow{\eta_1} & ko\langle 1\rangle & \xrightarrow{c_1} & \Sigma^2 ku & \xrightarrow{r} & \Sigma^2 ko \\ \uparrow & & \uparrow & & \uparrow & & \uparrow \\ \Sigma ko\langle 1\rangle & \xrightarrow{\eta_2} & ko\langle 2\rangle & \xrightarrow{c_2} & \Sigma^4 ku & \xrightarrow{r_1} & \Sigma^2 ko\langle 1\rangle \end{array}$$

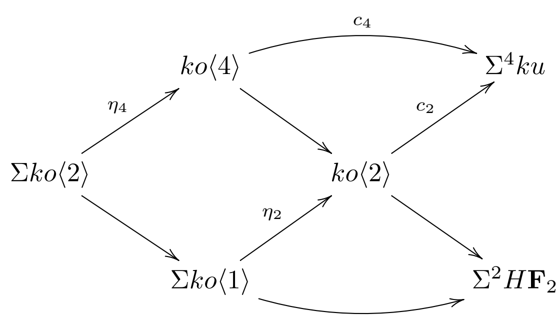
Again we need to record the maps induced in cohomology for use in the next step.

**Proposition 6.** $\Sigma ko\langle 1\rangle \xrightarrow{\eta_2} ko\langle 2\rangle \xrightarrow{c_2} \Sigma^4 ku$ *induces the short exact sequence*

$$\Sigma^2\mathcal{A}(1)/(Sq^2) \xleftarrow[\eta_2^*]{} \Sigma^2\mathcal{A}(1)/(Sq^3) \xleftarrow[c_2^*]{} \Sigma^4\mathcal{A}(1)/(Sq^1, Sq^3)$$



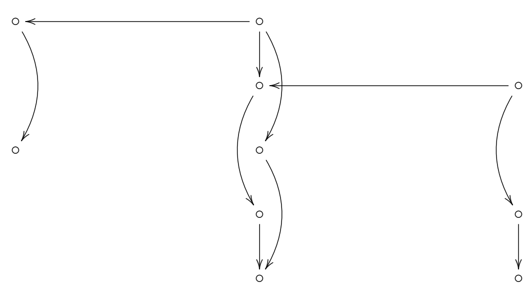
Now consider the braid of cofibrations induced by the composite $ko\langle 4\rangle \longrightarrow ko\langle 2\rangle \longrightarrow \Sigma^4 ku$.




Since $\eta_2^*$ is nonzero in degree 2, the map $\Sigma ko\langle 1\rangle \longrightarrow \Sigma^2 H\mathbf{F}_2$ is nontrivial, and hence the the fiber of $c_4$ is $\Sigma ko\langle 2\rangle$. Again, we need to record the maps induced in cohomology, and again, they 'roll' one step to the left.

**Proposition 7.** $\Sigma^3 ku \xrightarrow{r_2} \Sigma ko\langle 2\rangle \xrightarrow{\eta_4} ko\langle 4\rangle$ *induces the short exact sequence*

$$\Sigma^3\mathcal{A}(1)/(Sq^1, Sq^3) \xleftarrow[r_2^*]{} \Sigma^3\mathcal{A}(1)/(Sq^3) \xleftarrow[\eta_4^*]{} \Sigma^4\mathcal{A}(1)/(Sq^1, Sq^2Sq^3)$$

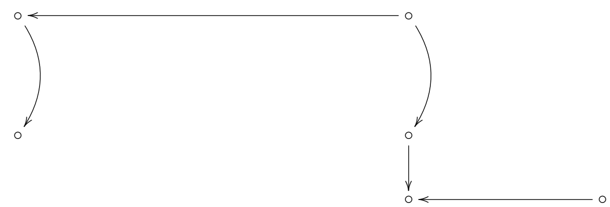


Since $\eta_4^*$ sends the generator to $Sq^1$, we get a map of cofiber sequences whose fiber gives the next lift, $\eta_8 c_8 r_4$.

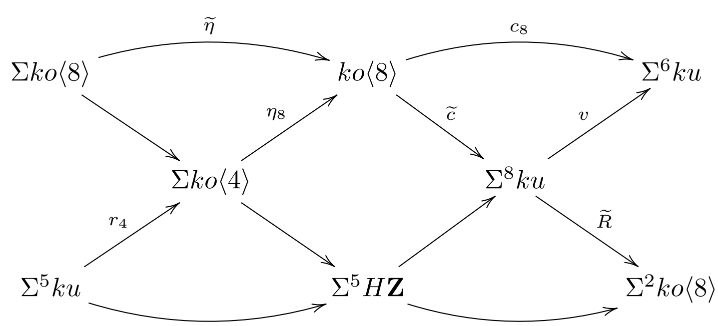
$$\begin{array}{ccccccc}
\Sigma^3 H\mathbf{Z} & \longrightarrow & \Sigma^3 H\mathbf{F}_2 & \xrightarrow[Sq^1]{} & \Sigma^4 H\mathbf{Z} & \longrightarrow & \Sigma^4 H\mathbf{Z} \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
\Sigma^3 ku & \xrightarrow{r_2} & \Sigma ko\langle 2\rangle & \xrightarrow{\eta_4} & ko\langle 4\rangle & \xrightarrow{c_4} & \Sigma^4 ku \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
\Sigma^5 ku & \xrightarrow{r_4} & \Sigma ko\langle 4\rangle & \xrightarrow{\eta_8} & ko\langle 8\rangle & \xrightarrow{c_8} & \Sigma^6 ku
\end{array}$$

**Proposition 8.** $\Sigma^5 ku \xrightarrow{r_4} \Sigma ko\langle 4\rangle \xrightarrow{\eta_8} ko\langle 8\rangle$ *induces the short exact sequence*

$$\Sigma^5\mathcal{A}(1)/(Sq^1, Sq^3) \xleftarrow[r_4^*]{} \Sigma^5\mathcal{A}(1)/(Sq^1, Sq^2Sq^3) \xleftarrow[\eta_8^*]{} \Sigma^8\mathbf{F}_2$$



Finally, consider the braid of cofibrations induced by the composite $\Sigma ko\langle 8\rangle \longrightarrow \Sigma ko\langle 4\rangle \xrightarrow{\eta_8} ko\langle 8\rangle$.




Since $r_4^*$ is an isomorphism on $H^5$, the map $\Sigma^5 ku \longrightarrow \Sigma^5 H\mathbf{Z}$ must be the bottom cohomology generator, justifying the appearance of $\Sigma^8 ku$ and $v$ in this braid.

The result is a cofiber sequence $\Sigma^9 ko \longrightarrow \Sigma^8 ko \longrightarrow \Sigma^8 ku$. The maps are $ko$-module maps by construction, and agree with the 8-fold suspensions of $\eta$, $c$ and $R$ in homotopy, by the maps $X\langle 8\rangle \longrightarrow X$. The adjunction $F_{ko}(\Sigma^9 ko, \Sigma^8 ko) \simeq F(S^9, \Sigma^8 ko)$, shows that a $ko$-module map $\Sigma^9 ko \longrightarrow \Sigma^8 ko$ is determined by its effect on homotopy. Therefore, the first map, and hence the other two, are the 8-fold suspensions of $\eta$, $c$ and $R$. □

## References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Department of Mathematics, Wayne State University, Detroit, Michigan 48202, USA
*E-mail address*: rrb@math.wayne.edu