# Characteristic Classes in K-Theory Connective K-theory of BG, G Compact Lie


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Topology Seminar
Universitetet i Bergen
10 August 2011

# Outline

For tori and 'symplectic tori' we have:

- $RU(T^n) = \mathbf{Z}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$
- $RU(Sp(1)^n) = \mathbf{Z}[s_1, \ldots, s_n]$

If $\lambda_i$ is the $i^{\text{th}}$ exterior power of the defining representation, then:

- $RU(Sp(n)) = \mathbf{Z}[\lambda_1, \ldots, \lambda_n]$
- $RU(SU(n)) = \mathbf{Z}[\lambda_1, \ldots, \lambda_{n-1}]$
- $RU(U(n)) = \mathbf{Z}[\lambda_1, \ldots, \lambda_n, \lambda_n^{-1}]$
- $RU(O(n)) = \mathbf{Z}[\lambda_1, \ldots, \lambda_n]/(\lambda_n^2 - 1, \lambda_i \lambda_n - \lambda_{n-i})$
- $RU(SO(2n+1)) = \mathbf{Z}[\lambda_1, \ldots, \lambda_n]$ with $\lambda_{n+i} = \lambda_{n+1-i}$
- $RU(SO(2n)) = \mathbf{Z}[\lambda_1, \ldots, \lambda_{n-1}, \lambda_n^+, \lambda_n^-]/R$
  with $\lambda_{n+i} = \lambda_{n-i}$, $\lambda_n = \lambda_n^+ + \lambda_n^-$ and
  $R = ((\lambda_n^+ + \sum_i \lambda_{n-2i})(\lambda_n^- + \sum_i \lambda_{n-2i}) - (\sum \lambda_{n-1-2i})^2)$

$$RU(T^n) = \mathbf{Z}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$$

All simple representations but the trivial one are complex.
The integral cohomology ring is

$$H^* BT^n = \mathbf{Z}[y_1, \ldots, y_n]$$

with $y_i = c_1(t_i)$.

### Theorem

$$ku^* BT^n = ku^*[[y_1, \ldots, y_n]]$$

*with* $y_i = c_1^{ku}(t_i)$ *and*

$$ku^*_{T^n} = \mathrm{MRees}(RU(T^n)) = \frac{ku^*[y_1, \overline{y}_1 \ldots, y_n, \overline{y}_n]}{(v y_i \overline{y}_i = y_i + \overline{y}_i)}$$

*with* $\overline{y}_i = c_1^{ku}(t_i^{-1}) \mapsto -y_i/(1 - v y_i)$.

The calculation of $ku^*_{T^n}$ uses the pullback square

$$\begin{array}{ccc} ku^*_{T^n} & \longrightarrow & KU^*_{T^n} \\ \downarrow & & \downarrow \\ ku^*BT^n & \longrightarrow & KU^*BT^n \end{array}$$

We have the relations $t_i = 1 - vy_i$ and $t_i^{-1} = 1 - v\overline{y}_i$, hence

$$KU^*_{T^n} = KU^*[t_1^{\pm 1}, \ldots, t_n^{\pm 1}] \;\cong\; KU^*[y_1, \overline{y}_1, \ldots, y_n, \overline{y}_n]/R,$$

where $R$ is the ideal generated by the relations $vy_i\overline{y}_i = y_i + \overline{y}_i$.
Then

$$ku^*_{T^n} = ku^*[[y_1, \ldots, y_n]] \cap KU^*[y_1, \overline{y}_1, \ldots, y_n, \overline{y}_n]/R$$

# Real connective K-theory

$H^*BT^n$ is free over $E[Sq^2] = H^*C\eta$, and $ko \wedge C\eta = ku$.
Hence, the Adams spectral sequence for $ko^*BT^n$ collapses and is concentrated in even degrees.
Hence $\eta$ acts trivially and the $\eta - c - R$ sequence is just a short exact sequence

$$0 \longrightarrow ko^*BT^n \xrightarrow{c} ku^*BT^n \xrightarrow{R} ko^{*+2}BT^n \longrightarrow 0$$

Complex conjugation acts by

- $\tau(v) = -v$
- $$\tau(y_i) = \tau(\frac{1-t_i}{v}) = \frac{1-t_i^{-1}}{-v} = -\overline{y}_i$$

The Bockstein differential $cR$ is then exact and $ko^*$-linear, hence $2v^2$ linear. But $ku^*BT^n$ is 2-torsion-free, so $cR$ is $v^2$-linear. Hence

$$v^2 cR(x) = cR(v^2 x) = cr(vx) = (1+\tau)(vx) = vx - v\tau(x)$$

or

$$cR(x) = \frac{x - \tau(x)}{v}$$

## Theorem

*$ko^*_{T^n} = (ku^*_{T^n})^{C_2}$ where $C_2$ acts by conjugation.*

## Proof.

$ko^*_{T^n} = \mathrm{im}(c) = \ker(R) = \ker(vcR) = \ker(1-\tau)$. □

$$RU(SU(n)) = \mathbf{Z}[\lambda_1, \ldots, \lambda_{n-1}].$$

We have $\overline{\lambda_i} = \lambda_{n-i}$, so the $\lambda_i$ are all complex unless $n = 2m$, when $\lambda_m$ is real if $m$ is even and quaternionic if $m$ is odd. The integral cohomology is

$$H^* BSU(n) = \mathbf{Z}[c_2, \ldots, c_n]$$

with $c_i = c_i(\lambda_1)$. The connective complex $K$-theory is easy to compute.

**Theorem**

*$ku^* BSU(n) = ku^*[[c_2, \ldots, c_n]]$ and*

$$ku^*_{SU(n)} = \mathrm{MRees}(RU(SU(n))) = ku^*[c_2, \ldots, c_n].$$

## Proof.

Since $H^*BSU(n)$ is concentrated in even degrees, the Atiyah-Hirzebruch spectral sequence implies $ku^*BSU(n)$ must be the complete $ku^*$ algebra freely generated by $c_2, \ldots, c_n$.

In $KU^*_{SU(n)}$, we have

$$\lambda_i = \sum_{j=0}^{i} (-1)^j \binom{n-j}{n-i} c_j^R = \sum_{j=0}^{i} (-1)^j \binom{n-j}{n-i} v^j c_j^{ku}.$$

and $\lambda_n = 1$.

Hence, the $c_j = c_j^{ku}$ generate, and $c_1 - vc_2 + \cdots + (-v)^{n-1} c_n = 0$. Thus $KU^*_{SU(n)}$ is polynomial on any $n-1$ of $c_1, \ldots, c_n$. In particular, $KU^*_{SU(n)} = KU^*[c_2, \ldots, c_n]$. □

## Proof.

(Cont.) The pullback square

$$\begin{array}{ccc} ku^*_{SU(n)} & \longrightarrow & KU^*[c_2, \ldots, c_n] \\ \downarrow & & \downarrow \\ ku^*[[c_2, \ldots, c_n]] & \longrightarrow & KU^*[[c_2, \ldots, c_n]] \end{array}$$

shows that $ku^*_{SU(n)} = ku^*[c_2, \ldots, c_n]$. □

$$RU(U(n)) = \mathbf{Z}[\lambda_1, \ldots, \lambda_n, \lambda_n^{-1}]$$

The integral cohomology is

$$H^* BU(n) = \mathbf{Z}[c_1, \ldots, c_n]$$

where $c_i = c_i(\lambda_1)$. Again, the complex connective $K$-theory follows immediately.

**Theorem**

*$ku^* BU(n) = ku^*[[c_1, \ldots, c_n]]$ and*

$$ku^*_{U(n)} = \mathrm{MRees}(RU(U(n))) = ku^*[c_1, \ldots, c_n, \Delta^{-1}]$$

*where $\Delta = \lambda_n = 1 - vc_1 + v^2 c_2 - \cdots + (-v)^n c_n$.*

**Proof.**

The argument is nearly the same as for $SU(n)$, except that $KU^*_{U(n)}$ is not polynomial, but is instead $KU^*[c_1, \ldots, c_n, \Delta^{-1}]$. □

In cohomology, restriction along the inclusion $SU(n) \longrightarrow U(n)$ is the quotient which sends $c_1$ to 0. The proper way to think of this is that we are taking the quotient by the determinant of the defining representation. In $K$-theory, the map this induces is more interesting.

## Theorem

*The restriction homomorphism $ku^*_{U(n)} \longrightarrow ku^*_{SU(n)}$ is the quotient $ku^*[c_1, \ldots, c_n, \Delta^{-1}] \longrightarrow ku^*[c_2, \ldots, c_n]$ which sends $\Delta$ to 1 and $c_1$ to $vc_2 - v^2c_3 + \cdots - (-v)^{n-1}c_n$.*

## Proof.

$SU(n)$ is the kernel of the determinant $U(n) \longrightarrow U(1)$. The determinant sends $y = (1 - \lambda_1)/v \in ku^2_{U(1)}$ to $(1 - \lambda_n)/v = c_1 - vc_2 + v^2c_3 - \cdots$, so this must go to zero in $ku^*_{SU(n)}$. After dividing by this, we have an isomorphism, by the calculation of $ku^*_{SU(n)}$. □

Consider the conjugate Chern classes $\overline{c}_i(V) = c_i(\overline{V})$.

## Proposition

*Restriction* $ku^*_{U(n)} \longrightarrow ku^*_{T(n)}$ *sends* $\overline{c}_i$ *to* $\sigma_i(\overline{y}_1, \ldots, \overline{y}_n)$. *There is a relation*

$$\Delta \overline{c}_i = \sum_{k=i}^{n} (-1)^k \binom{k}{i} v^{k-i} c_k$$

The conjugate $\overline{\Delta} = \overline{\lambda}_n = 1 - v\overline{c}_1 + v^2\overline{c}_2 - \cdots \pm v^n \overline{c}_n$ satisfies $\Delta\overline{\Delta} = 1$. Collecting terms we find

## Proposition

*In* $ku^*_{U(n)}$,

$$c_1 + \overline{c}_1 = -\sum_{k=2}^{2n} (-v)^{k-1} \sum_{i+j=k} c_i \overline{c}_j$$

□

$$RU(Sp(n)) = \mathbf{Z}[\lambda_1, \ldots, \lambda_n]$$

The $\lambda_{2i}$ are real and the $\lambda_{2i+1}$ are quaternionic, hence all are self conjugate.
Note that $\lambda_1 = \mathbf{H}^n = \mathbf{C}^{2n}$, which is $2n$ dimensional, but its higher exterior powers $\lambda_{n+1}, \ldots \lambda_{2n}$ can be expressed in terms of the first $n$.

The integral cohomology is

$$H^* BSp(n) = \mathbf{Z}[p_1, \ldots, p_n]$$

with $|p_i| = 4i$.

Restriction along $Sp(1)^n \longrightarrow Sp(n)$ will play much the same role for $Sp(n)$ as restriction along $T^n = U(1)^n \longrightarrow U(n)$ plays for $U(n)$, so we start by considering $Sp(1)^n$.

$$RU(Sp(1)^n) = \mathbf{Z}[s_1, \ldots, s_n]$$

where

$$s_i : Sp(1)^n \xrightarrow{p_i} Sp(1) \;\cong\; SU(2) \subset U(2)$$

Hence $c_1^{ku}(s_i) = vc_2^{ku}(s_i)$ and

$$v^2 c_2^{ku}(s_i) = vc_1^{ku}(s_i) = c_2^R(s_i) = c_1^R(s_i) = 2 - s_i.$$

Thus, we have classes $z_i = c_2^{ku}(s_i) \in ku^4(BSp(1)^n)$ which satisfy

$$v^2 z_i = 2 - s_i.$$

We will see that $z_i$ comes from $ko^*$. The integral cohomology ring is

$$H^* BSp(1)^n = \mathbf{Z}[z_1, \ldots, z_n]$$

with $z_i = p_1(s_i)$, the first Pontrjagin class of $s_i$.

## Theorem

*There are compatible generators $z_i$ so that*

- $ku^* BSp(1)^n = ku^*[[z_1, \ldots, z_n]]$
- $ko^* BSp(1)^n = ko^*[[z_1, \ldots, z_n]]$
- $ku^*_{Sp(1)^n} = ku^*[z_1, \ldots, z_n] = \mathrm{MRees}(RU(Sp(1)^n))$
- $ko^*_{Sp(1)^n} = ko^*[z_1, \ldots, z_n]$

*In particular, $z_i^{ku} \in ku^4_{Sp(1)^n}$ and $z_i^{ko} \in ko^4_{Sp(1)^n}$ satisfy*

- $v^2 z_i^{ku} = 2 - s_i \in ku^0_{Sp(1)^n} = RU(Sp(1)^n)$,
- $\alpha z_i^{ko} = 2(2 - s_i) \in ko^0_{Sp(1)^n} = RO(Sp(1)^n)$ *and*
- $\beta z_i^{ko} = 2 - s_i \in ko^{-4}_{Sp(1)^n} = RSp(Sp(1)^n)$.

## Proof.

The Adams spectral sequence collapses at

$$E_2^{*,*} = H^*BSp(1)^n \otimes \mathrm{Ext}_{\mathcal{A}(1)}^{*,*}(\mathbf{F}_2, \mathbf{F}_2) \Longrightarrow ko^*BSP(1)^n$$

and similarly for $E(1)$ and $ku^*$.

The equivariant cases then follow by the defining pullback squares

$$\begin{array}{ccc} ku^*_{Sp(1)^n} & \longrightarrow & KU^*[z_1, \ldots, z_n] \\ \downarrow & & \downarrow \\ ku^*[[z_1, \ldots, z_n]] & \longrightarrow & KU^*[[z_1, \ldots, z_n]] \end{array}$$

The periodic groups are as claimed because we can change generators from the $s_i$ to the $z_i = (2 - s_i)/v^2$. This is MRees$(RU(Sp(1)^n))$: all irreducible representations are two dimensional, so $JU_{2n} = JU_{2n-1} = (JU_2)^n$. □

# Pontrjagin classes

## Definition

The $k^{\text{th}}$ representation theoretic *Pontrjagin class* of an $n$-dimensional symplectic representation $V : G \longrightarrow Sp(n)$ is

$$p_k^R(V) = \sum_{j=0}^{k} (-1)^j 2^{k-j} \binom{n-j}{n-k} \Lambda^j(V)$$

## Proposition

*The restriction $RU(Sp(n)) \longrightarrow RU(Sp(1)^n)$ sends $p_k^R$ to $\sigma_k(2-s_1,\ldots,2-s_n)$. The representation $p_k^R$ is real if $k$ is even, and quaternionic if $k$ is odd.*

Accordingly, we shall generally consider $p_{2i}^R$ as an element of $RO(G)$ and $p_{2i+1}^R$ as an element of $RSp(G)$. Note, however, that representations which are not irreducible can be both real and quaternionic.

## Theorem

*We have*

- $ku^* BSp(n) = ku^*[[p_1, \ldots, p_n]]$
- $ko^* BSp(n) = ko^*[[p_1, \ldots, p_n]]$
- $ku^*_{Sp(n)} = ku^*[p_1, \ldots, p_n]$
- $ko^*_{Sp(n)} = ko^*[p_1, \ldots, p_n]$.

*In each case,* $p_k$ *restricts to* $\sigma_k(z_1, \ldots, z_n)$.

*In* $ku^*$, $v^{2k} p_k^{ku} = p_k^R \in ku^0_{Sp(n)} = RU(Sp(n))$.

*In* $ko^*$, $\beta^k p_{2k}^{ko} = p_{2k}^R \in ko^0_{Sp(n)} = RO(Sp(n))$ *and* $\beta^k p_{2k+1}^{ko} = p_{2k+1}^R \in ko^4_{Sp(n)} = JSp(Sp(n))$.

## Definition

Let $V : G \longrightarrow Sp(n)$ be a symplectic representation. For $E = RU$, $ko$, $KO$, $ku$, $KU$ or $H$, we define the *Pontrjagin class* $p_i^E(V) \in E_G^{4i}$ to be $V^*(p_i)$. It is convenient to collect these into the *total Pontrjagin class*

$$p_\bullet^E(V) = 1 + p_1^E(V) + p_2^E(V) + \cdots + p_n^E(V)$$

and to let $p_i^E(V) = 0$ if $i > n$.

## Corollary

$$p_\bullet^E(V \oplus W) = p_\bullet^E(V) p_\bullet^E(W)$$

## Lemma

*The restriction* $ku^*_{Sp(1)^n} \longrightarrow ku^*_{T(n)}$ *maps* $z_i$ *to* $y_i\overline{y}_i$.

Write $\overline{c}_i(V) = c_i(\overline{V})$ for the Chern classes of the complex conjugate of a representation.

## Theorem

*The restriction maps* $ku^*_{U(2n)} \xrightarrow{q^*} ku^*_{Sp(n)} \xrightarrow{\widetilde{c}^*} ku^*_{U(n)}$ *obey*

$$c_k \mapsto \sum_{0 \le 2i \le k} \binom{k-i}{i} v^{k-2i} p_{k-i} \mapsto \sum_{i+j=k} c_i \overline{c}_j .$$

Specializing to ordinary cohomology by setting $v = 0$ we obtain the usual relations (up to sign):

$$q^*(p_n^H) = \sum_{i+j=2n} c_i^H \overline{c}_j^H = \sum_{i+j=2n} (-1)^j c_i^H c_j^H,$$

$$\widetilde{c}^*(c_{2i-1}) = 0, \qquad \text{and} \qquad \widetilde{c}^*(c_{2i}) = p_i .$$

For $Sp(4)$, for example,

$$
\begin{array}{rlllllll}
c_1 & \mapsto & vp_1 & & & & \mapsto & c_1 + \overline{c}_1 \\
c_2 & \mapsto & p_1 & +v^2p_2 & & & \mapsto & c_2 + c_1\overline{c}_1 + \overline{c}_2 \\
c_3 & \mapsto & & 2vp_2 & +v^3p_3 & & \mapsto & c_3 + c_2\overline{c}_1 + c_1\overline{c}_2 + \overline{c}_3 \\
c_4 & \mapsto & & p_2 & +3v^2p_3 & +v^4p_4 & \mapsto & c_4 + c_3\overline{c}_1 + c_2\overline{c}_2 + c_1\overline{c}_3 + \overline{c}_4 \\
c_5 & \mapsto & & & 3vp_3 & +4v^3p_4 & \mapsto & c_4\overline{c}_1 + c_3\overline{c}_2 + c_2\overline{c}_3 + c_1\overline{c}_4 \\
c_6 & \mapsto & & & p_3 & +6v^2p_4 & \mapsto & c_4\overline{c}_2 + c_3\overline{c}_3 + c_2\overline{c}_4 \\
c_7 & \mapsto & & & & 4vp_4 & \mapsto & c_4\overline{c}_3 + c_3\overline{c}_4 \\
c_8 & \mapsto & & & & p_4 & \mapsto & c_4\overline{c}_4
\end{array}
$$

It is more difficult to get good expressions for the images of the individual $p_i$. However, for $i = 1$, using the fact that $v$ acts monomorphically on $ku^*_{U(n)}$ we have

$$p_1 \mapsto \frac{c_1 + \overline{c}_1}{v} = \sum_{k=2}^{n} (-v)^{k-2} \sum_{i+j=k} c_i \overline{c}_j$$

$$= c_2 + c_1\overline{c}_1 + \overline{c}_2 - v \sum_{k=3}^{n} (-v)^{k-3} \sum_{i+j=k} c_i \overline{c}_j.$$

In cohomology, where $v = 0$ and $\overline{c}_i = (-1)^i c_i$, we have

$$p_1 \mapsto c_2 + c_1\overline{c}_1 + \overline{c}_2 = 2c_2 - c_1^2$$

with our normalization of the $p_i$. Thus, if $c_1 = 0$, then $c_2 = p_1/2$.

Finally, we provide the following *symplectic splitting principle*.

## Theorem

*Let $\xi$ be an $Sp(n)$ bundle over $X$. Then there exists a map $f : Y \longrightarrow X$ such that $f^*\xi$ is a sum of symplectic line bundles and $f^* : H^*X \longrightarrow H^*Y$ is a monomorphism.*

## Proof.

*Let $Y$ be the pullback*

$$\begin{array}{ccc} Y & \longrightarrow & BSp(1)^n \\ \downarrow{\scriptstyle f} & & \downarrow \\ X & \longrightarrow & BSp(n) \end{array}$$

*along the classifying map of the bundle $\xi$. The universal bundle over $BSp(n)$ splits as a sum of line bundles over $BSp(1)^n$, so $f^*\xi$ also splits in this manner.*

*The Serre spectral sequence gives the cohomology statement.* □

$$RU(O(n)) = \mathbf{Z}[\lambda_1, \ldots, \lambda_n]/(\lambda_n^2 - 1, \lambda_i\lambda_n - \lambda_{n-i}).$$

These representations are all real, so that complexification and quaternionification are isomorphisms

$$RO(O(n)) \xrightarrow{\cong} RU(O(n)) \xrightarrow{\cong} RSp(O(n)).$$

The integral cohomology is complicated. The best approach is to give the mod 2 cohomology, and if integral issues matter, the cohomology localized away from 2. We have

$$H\mathbf{F}_2^* BO(n) = \mathbf{F}_2[w_1, \ldots, w_n]$$

where $w_i = w_i(\lambda_1)$.

Rewrite the representation ring in terms of Chern classes, as usual: let $c_i = c_i^K(\lambda_1) \in K^{2i}_{O(n)}$, so that

$$\lambda_i = \sum_{j=0}^{i} (-1)^j \binom{n-j}{n-i} v^j c_j.$$

Rather than replace $\lambda_n$ by the top Chern class, $c_n$, we use the first Chern class of the determinant representation, $c = c_1^K(\lambda_n) \in K^2_{O(n)}$. This satisfies $vc = 1 - \lambda_n$, which is much more convenient than $v^n c_n = 1 - \lambda_1 + \cdots + (-1)^n \lambda_n$.

## Proposition

$$KU^*_{O(2n+1)} = KU^*[c_1, \ldots, c_n, c]/(vc^2 - 2c)$$

*and*

$$KU^*_{O(2n)} = KU^*[c_1, \ldots, c_n, c]/(vc^2 - 2c, c \sum_{i=0}^{n} \binom{2n-i}{n} (-v)^i c_i)$$

To compute $ku^*BO(n)$, we need to determine the $E(1)$-module structure of $H\mathbf{F}_2^*BO(n)$. We start with its stable type. Let $\epsilon$ be 0 or 1.
First, the submodule

$$\mathbf{F}_2[w_2^2, w_4^2, \ldots, w_{2n}^2] \longrightarrow H^*BO(2n+\epsilon)$$

is a trivial $E(1)$-submodule.
Second, the reduced homology of $BO(1)$ is the ideal $(w_1)$ in $\mathbf{F}_2[w_1]$, and as an $E(1)$-submodule,

$$(w_1) \otimes \mathbf{F}_2[w_2^2, w_4^2, \ldots, w_{2n-2}^2] \longrightarrow H^*BO(2n-\epsilon)$$

is a direct sums of suspensions of $(w_1)$.
The sum of these two submodules exhausts the 'interesting' part of $H^*BO(n)$, in the sense that the complementary summand is $E(1)$-free.

## Theorem

*The inclusions*

$$\mathbf{F}_2[w_2^2, w_4^2, \ldots, w_{2n}^2] \oplus (w_1) \otimes \mathbf{F}_2[w_2^2, w_4^2, \ldots, w_{2n-2}^2] \longrightarrow H^*BO(2n)$$

*and*

$$\mathbf{F}_2[w_2^2, w_4^2, \ldots, w_{2n}^2] \oplus (w_1) \otimes \mathbf{F}_2[w_2^2, w_4^2, \ldots, w_{2n}^2] \longrightarrow H^*BO(2n+1)$$

*induce isomorphisms in $Q_0$ and $Q_1$ homology.*

## Corollary

*As an $E[Q_0, Q_1]$-module, $H^*BO(n)$ is the sum of trivial modules, suspensions of $H^*BO(1)$, and free modules.*

## Proof.

The corollary follows by the result of Adams and Margolis, that $Q_0$ and $Q_1$ homology detects the stable isomorphism type of the module. □

In principle, this describes $H^*BO(n)$ as an $E(1)$-module but finding a good parametrization of the complementary $E(1)$-free submodule is non-trivial.
The $\mathcal{A}(1)$-module structure is not as simple, as $Sq^2$ does not annihilate all squares.
The $w_{2i}^2$ detect Pontrjagin classes $p_i$ of the defining representation and $w_1^2$ in the $(w_1)$ summand detects the first Chern class of the determinant representation.

## Corollary

*The Adams spectral sequence converging to $ku^*BO(n)$ collapses at $E_2$, and the natural homomorphism*

$$ku^*BO(n) \longrightarrow H^*BO(n) \oplus KU^*BO(n)$$

*is a monomorphism.*

# Comments on the proof

The $H(-, Q_0)$ isomorphism is straightforward, but the $H(-, Q_1)$ isomorphism requires a careful choice of generators.

Once the correct generators are identified, it turns out that the general case is just a regraded version of $H^*BO(4)$ tensored with an $E(1)$-trivial subalgebra.

See the book with Greenlees for details.

# $O(1)$

Recall that $ku^*_{O(1)} = ku^*[c]/(vc^2 - 2c)$ by the pullback square

$$\begin{array}{ccc} ku^*_{O(1)} & \longrightarrow & KU^*_{O(1)} = KU^*[c]/(vc^2 - 2c) \\ \downarrow & & \downarrow \\ ku^*BO(1) = ku^*[[c]]/(vc^2 - 2c) & \longrightarrow & KU^*BO(1) = KU^*[[c]]/(vc^2 - 2c) \end{array}$$

The Bockstein spectral sequence then gives

## Theorem

*There are unique elements $p_0 \in ko^0_{O(1)}$ and $p_1 \in ko^4_{O(1)}$ with complexifications $c(p_0) = vc$ and $c(p_1) = c^2$. The ring*

$$ko^*_{O(1)} = \frac{ko^*[p_0, p_1]}{(\eta p_1, \alpha p_1 - 4p_0, \beta p_1 - \alpha p_0, p_0 p_1 - 2p_1, p_0^2 - 2p_0)}$$

In terms of representation theory, this can be written as follows.

## Corollary

*$O(1)$-equivariant connective real K-theory has coefficient ring*

$$ko^i_{O(1)} = \begin{cases} RSp & i = -8k-4 \leq 0 \\ RO/2 & i = -8k-2 \leq 0 \\ RO/2 & i = -8k-1 \leq 0 \\ RO & i = -8k \leq 0 \\ JSp_k = JSp^k & i = 4k > 0 \\ 0 & \text{otherwise} \end{cases}$$

To justify the notation $p_i$:

## Theorem

*The restriction $ku^*_{Sp(1)} \longrightarrow ku^*_{O(1)}$ is:*

$$z = p_1(\lambda_1) \mapsto p_1, \qquad \alpha z \mapsto 4p_0, \qquad \text{and} \qquad \beta z \mapsto \alpha p_0,$$

**Proof.**

That $z$ maps to $p_1$ is evident by comparison with $ku^*$. The rest follows by the relations in $ko^*_{O(1)}$. □

Thus, $p_1$ really is the first Pontrjagin class of the quaternionic representation induced up from the defining representation of $O(1)$, while $p_0$ is a genuinely real class. We call it $p_0$ because of the relations which tie it to $p_1$.

# $O(2)$

## Corollary

$$
\begin{aligned}
KU^*_{O(2)} &= KU^*[c, c_1]/(vc^2 - 2c, c(2 - vc_1)) \\
&= KU^*[c, c_2]/(vc^2 - 2c, cc_2)
\end{aligned}
$$

## Proof.

The calculation

$$
\begin{aligned}
v^2 c_2 &= 1 - \lambda_1 + \lambda_2 \\
&= 1 - (2 - vc_1) + (1 - vc) \\
&= v(c_1 - c)
\end{aligned}
$$

shows that $c_1 = c + vc_2$. Then the relation $0 = c(2 - vc_1)$ becomes $v^2 cc_2 = 0$ since $c(2 - vc) = 0$. □

The connective $K$-theory is similar but somewhat larger.

**Theorem**

$ku^*_{O(2)} = ku^*[c, c_2]/(vc^2 - 2c, 2cc_2, vcc_2)$

**Proof.**

Decomposing $H^*BO(2)$ as an $E[Q_0, Q_1]$-module shows that $c$ and $c_2$ are algebra generators for $ku^*BO(2)$. The monomorphism into $H^*BO(2) \oplus KU^*BO(2)$ then shows the relations are complete. The pullback square then gives us $ku^*_{O(2)}$. □

The Bockstein spectral sequence then gives

## Theorem

$KO^*_{O(2)} = KO^*[p_0, r_0]/(p_0^2 - 2p_0, p_0 r_0)$ *and*

$$ko^*_{O(2)} = ko^*[p_0, p_1, p_2, r_0, r_1, s]/I$$

*where* $I$ *is the ideal generated by the relations*

| | | |
|---|---|---|
| $\eta p_1 = 0$ | $\alpha p_1 = 4p_0$ | $\beta p_1 = \alpha p_0$ |
| $\eta r_1 = 0$ | $\alpha r_1 = 4r_0$ | $\beta r_1 = \alpha r_0$ |
| $\eta s = 0$ | $\alpha s = 0$ | $\beta s = \eta^2 r_0$ |
| $p_0 p_2 = 0$ | $p_1 p_2 = s^2$ | |

| | $p_0$ | $p_1$ | $r_0$ | $r_1$ | $s$ |
|---|---|---|---|---|---|
| $p_0$ | $2p_0$ | $2p_1$ | $0$ | $0$ | $0$ |
| $p_1$ | $2p_1$ | $p_1^2$ | $0$ | $0$ | $p_1 s$ |
| $r_0$ | $0$ | $0$ | $\beta p_2$ | $\alpha p_2$ | $\eta^2 p_2$ |
| $r_1$ | $0$ | $0$ | $\alpha p_2$ | $4p_2$ | $0$ |

$p_0$ and $r_0$ are $1 - \det$ and the Euler class of the defining representation, respectively.

$p_1$ and $r_1$ are their images in $JSp = ko^4$. This explains the similarity of the action of $ko^*$ on them.

The class $p_2$ refines the square of the Euler class in the sense that $r_1^2 = 4p_2$, $r_0 r_1 = \alpha p_2$ and $r_0^2 = \beta p_2$.

The class $s$ is a square root of the product $p_1 p_2 = s^2$.

The relation $\beta s = \eta^2 r_0$ is hidden in the Bockstein spectral sequence. Representation theory (i.e., the map into $KO^*_{O(2)}$) and the Adams spectral sequence each work to recover it.

# $O(3)$

## Corollary

$KU^*_{O(3)} = KU^*[c, c_1]/(vc^2 - 2c)$

## Proposition

*$KO^*_{O(3)} = KO^*[p_0, q_0]/(p_0^2 - 2p_0)$ where $p_0$ and $q_0$ have complexifications $vc$ and $vc_1$ respectively. The restriction $KO^*_{O(3)} \longrightarrow KO^*_{O(2)}$ sends $p_0$ to $p_0$ and $q_0$ to $p_0 + r_0$.*

The Chern classes no longer suffice to generate $ku^*_{O(n)}$ for $n > 2$. Let $\overline{Q}_0 : H \longrightarrow \Sigma H\mathbf{Z}$ and $\overline{Q}_1 : H\mathbf{Z} \longrightarrow \Sigma^3 ku$ be the boundary maps in the cofiber sequences for $2 : H\mathbf{Z} \longrightarrow H\mathbf{Z}$ and $v : \Sigma^2 ku \longrightarrow ku$. They are lifts of the Milnor primitives $Q_0$ and $Q_1$.

## Definition

Let $q_2 = \overline{Q}_1\overline{Q}_0(w_2) \in ku^6BO(3)$ and $q_3 = \overline{Q}_1\overline{Q}_0(w_3) \in ku^7BO(3)$.

## Proposition

*The classes $q_2$ and $q_3$ are nonzero classes annihilated by $(2, v)$. The class $q_3$ is independent of $c$, $c_2$, and $c_3$, while $q_2 = cc_2 - 3c_3$. These are the only nonzero 2 or $v$-torsion classes in $ku^6BO(3)$ and $ku^7BO(3)$.*

## Theorem

*$ku^*BO(3) = ku^*[c, c_2, c_3, q_3]/R$, where $R$ is an ideal containing $(vc^2 - 2c, 2(cc_2 - 3c_3), v(cc_2 - 3c_3), 2q_3, vq_3, vcc_3 - 2c_3)$.*

# $O(n)$ for larger $n$

The free summands in $H^*BO(n)$ begin to get more complicated at $n = 4$. Let us write $w_S$ for the product $\prod_{i \in S} w_i$.

## Proposition

*Maximal $E(1)$-free summands of $H^*BO(n)$ are:*

$n = 4$

$$\mathbf{F}_2[w_1^2, w_2^2, w_3^2, w_4^2]\langle w_2, w_3, w_4, w_{234}\rangle$$
$$\oplus \mathbf{F}_2[w_1^2, w_2^2, w_4^2]\langle w_{24}\rangle$$

$n = 5$

$$\mathbf{F}_2[w_1^2, w_2^2, w_3^2, w_4^2, w_5^2]\langle w_2, w_3, w_4, w_5, w_{234}, w_{235}, w_{245}, w_{345}\rangle$$
$$\oplus \mathbf{F}_2[w_1^2, w_2^2, w_4^2, w_5^2]\langle w_{24}, w_{34}\rangle$$

$n = 6$

$$\mathbf{F}_2[w_1^2, w_2^2, w_3^2, w_4^2, w_5^2, w_6^2]\langle w_2, w_3, w_4, w_5, w_6, w_{26}, w_{234}, w_{235}, w$$
$$\oplus \mathbf{F}_2[w_1^2, w_2^2, w_3^2, w_4^2, w_5^2, w_6^2]\langle w_{236}, w_{246}, w_{256}, w_{346}, w_{456}, w_{23456}$$
$$\oplus \mathbf{F}_2[w_1^2, w_2^2, w_4^2, w_5^2, w_6^2]\langle w_{24}, w_{34}, w_{2456}\rangle$$
$$\oplus \mathbf{F}_2[w_1^2, w_2^2, w_4^2, w_6^2]\langle w_{46}\rangle$$

**Remark**

Each $w_S$ generating a free $E(1)$ will give rise to a $(2, v)$-annihilated class $\overline{Q}_1\overline{Q}_0(w_S) \in ku^*BO(n)$.

$$RU(SO(2n+1)) = \mathbf{Z}[\lambda_1, \ldots, \lambda_n]$$

with $\lambda_{n+i} = \lambda_{n+1-i}$ and

$$RU(SO(2n)) = \mathbf{Z}[\lambda_1, \ldots, \lambda_{n-1}, \lambda_n^+, \lambda_n^-]/R$$

with $\lambda_{n+i} = \lambda_{n-i}$ and $\lambda_n = \lambda_n^+ + \lambda_n^-$. The ideal $R$ is generated by one relation

$$(\lambda_n^+ + \sum_i \lambda_{n-2i})(\lambda_n^- + \sum_i \lambda_{n-2i}) = (\sum \lambda_{n-1-2i})^2$$

All the $\lambda_i$ are real. $RU(SO(2n))$ is free over $RU(SO(2n+1))$ on $\{1, \lambda_n^+\}$.

$H^*BSO(n) = \mathbf{F}_2[w_2, \ldots, w_n]$ where $w_i = w_i(\lambda_1)$.
We have already examined $SO(2) = T(1)$ and found (writing $c_1$ rather than $y_1$ here)

$$ku^*_{SO(2)} = ku^*[c_1, \overline{c}_1]/(vc_1\overline{c}_1 = c_1 + \overline{c}_1)$$

and

$$ku^*BSO(2) = ku^*[[c_1]].$$

The maps induced in $ku^*$ by the fibre sequence $SO(2) \xrightarrow{i} O(2) \xrightarrow{\det} O(1)$ are

## Proposition

$\det^*(c) = c$, *while* $i^*(c) = 0$, $i^*(c_2) = c_1\overline{c}_1$ *and* $i^*(c_1) = i^*(c + vc_2) = c_1 + \overline{c}_1$.

# $SO(3)$

$RU(SO(3)) = \mathbf{Z}[\lambda_1]$ with $\lambda_2 = \lambda_1$ and $\lambda_3 = 1$.

## Proposition

$KU^*_{SO(3)} = KU^*[c_2]$

## Proof.

The Chern classes of the defining representation of $SO(3)$ satisfy $c_1 = vc_2$, $c_3 = 0$ and $v^2c_2 = vc_1 = 3 - \lambda_1$. □

## Theorem

$$ku^*_{SO(3)} = ku^*[c_2, c_3]/(2c_3, vc_3).$$

*The first Chern class, $c_1 = vc_2$. The restriction $ku^*_{O(3)} \longrightarrow ku^*_{SO(3)}$ sends $c$ and $q_3$ to 0, and sends each $c_i$ to $c_i$.*

**Proof.**

The Adams spectral sequence again collapses and gives us a monomorphism into the sum of mod 2 cohomology and periodic K-theory. This makes it easy to show $ku^*BSO(3) = ku^*[[c_2, c_3]]/(2c_3, vc_3)$. The pullback square then gives $ku^*_{SO(3)} = ku^*[c_2, c_3]/(2c_3, vc_3)$. □

In general, we expect $c_1 = vc_2 - v^2c_3$, since this is true in $SU(n)$, but here $v^2c_3 = 0$.

The restriction from $O(3)$ is computed by using the monomorphism to periodic $K$-theory plus mod 2 cohomology. Note that $q_2 = cc_2 - 3c_3$ restricts to $c_3$.

Thank you