THE TANGENT BUNDLE OF $\mathbb{RP}^n$

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1. Real case

Milnor and Stasheff [1, Thm 4.5] show that the tangent bundle of real projective space, $\mathrm{T}\,\mathbb{RP}^n$ satisfies

$$\mathrm{T}\,\mathbb{RP}^n \oplus \epsilon = (n+1)\ell$$

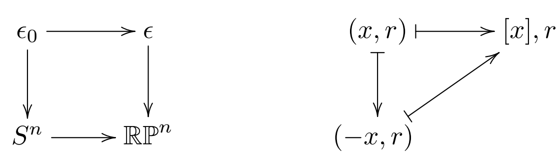
where ϵ is the trivial line bundle and ℓ is the non-trivial line bundle over $\mathbb{RP}^n$. Recall that we may take ℓ to be the tautological, or Hopf, bundle

$$\ell = \left\{([x], v) \in \mathbb{RP}^n \times \mathbb{R}^{n+1} \mid v \in [x]\right\} \longrightarrow \mathbb{RP}^n$$

given by projection onto the first factor. They use an indirect argument involving the bundle of homomorphisms from ℓ to the complement of ℓ inside the trivial $(n+1)$-dimensional bundle.

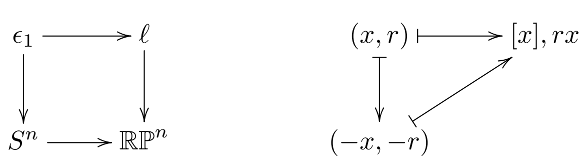
By pulling back to S^n, we write down an immediate explicit isomorphism which is covered by a C_2-equivariant isomorphism of bundles over the C_2 space S^n with the antipodal action.

Step 1: The trivial bundle ϵ over $\mathbb{RP}^n$ pulls back to the trivial bundle with the trivial action over S^n:



On the right we exhibit the C_2-action on the total space of the bundle over S^n together with the map of total spaces.

Step 2: The non-trivial line bundle ℓ over $\mathbb{RP}^n$ pulls back to the trivial bundle with the non-trivial action over S^n:



Step 3: We identify $\mathrm{T}\,S^n$ with its image

$$\left\{(x, v) \in R^{n+1} \times \mathbb{R}^{n+1} \mid x \cdot x = 1, x \cdot v = 0\right\}$$

in the tangent bundle of $\mathbb{R}^{n+1}$. Then $\mathrm{T}\,\mathbb{RP}^n$ can be viewed as equivalence classes of such pairs: $[x, v] = \{(x, v), (-x, -v)\}$. There is an evident map of tangent bundles:

$$\begin{array}{ccc} \mathrm{T}\,S^n & \longrightarrow & \mathrm{T}\,\mathbb{RP}^n \\ \downarrow & & \downarrow \\ S^n & \longrightarrow & \mathbb{RP}^n \end{array} \qquad \begin{array}{ccc} (x, v) & \longmapsto & [x, v] \\ \downarrow & \nearrow & \\ (-x, -v) & & \end{array}$$

Step 4: We have the classic stable trivialization of the tangent bundle of S^n with its equivariance made clear:

$$\begin{array}{ccc} \mathrm{T}\,S^n \oplus \epsilon_0 & \longrightarrow & (n+1)\epsilon_1 \\ \downarrow & & \downarrow \\ S^n & = & S^n \end{array} \qquad \begin{array}{ccc} (x, v, r) & \longmapsto & (x, v + rx) \\ \downarrow & & \downarrow \\ (-x, -v, r) & \longmapsto & (-x, -v - rx) \end{array}$$

Now we can assemble these into the isomorphism we require: the dotted map in the diagram below.

$$\begin{array}{ccc} \mathrm{T}\,S^n \oplus \epsilon_0 & \longrightarrow & \mathrm{T}\,\mathbb{RP}^n \oplus \epsilon \\ \downarrow & & \vdots \\ (n+1)\epsilon_1 & \longrightarrow & (n+1)\ell \end{array} \qquad \begin{array}{ccc} (x, v, r) & \longmapsto & [x, v], r \\ \downarrow & & \vdots \\ (x, v + rx) & \longmapsto & [x], ((v_i + rx_i)_0^n x) \end{array}$$

The last coordinate $((v_i + rx_i)_0^n x)$ is the $(n + 1)$-tuple of multiples of x,

$$(v_0 + rx_0)x, (v_1 + rx_1)x, \ldots, (v_n + rx_n)x).$$

2. Complex case

We pull back the bundles from $\mathbb{CP}^n$ along the Hopf map $p : S^{2n+1} \longrightarrow \mathbb{CP}^n$ to get S^1-equivariant bundles over S^{2n+1}. We again let ℓ be the tautological (Hopf) bundle

$$\ell = \big\{([z], v) \in \mathbb{CP}^n \times \mathbb{C}^{n+1} \mid v \in [z]\big\} \longrightarrow \mathbb{CP}^n$$

given by projection onto the first factor. Inspecting transition functions, we see that

$$\ell^{-1} = \mathbb{CP}^{n+1} - \{[0, \ldots, 0, 1]\} \longrightarrow \mathbb{CP}^n,$$

given by the projection $[z_0, \ldots, z_{n+1}] \mapsto [z_0, \ldots, z_n]$. Evidently, ℓ^{-1} is the normal bundle of the inclusion $\mathbb{CP}^n \longrightarrow \mathbb{CP}^{n+1}$.

In the real case these are isomorphic bundles: $\mathrm{Pic}(\mathbb{RP}^n) = \mathbb{Z}/2 = \{\epsilon, \ell\}$. In the complex case, we have $\mathrm{Pic}(\mathbb{CP}^n) = \mathbb{Z} = \langle \ell \rangle$.

Generalizing the isomorphism of the preceding section from the real case to the complex case requires that we determine which power of ℓ is relevant. Equivariance is a good guide.

We will write $\epsilon \longrightarrow \mathbb{CP}^n$ for the trivial line bundle, and $\epsilon_i \longrightarrow S^{2n+1}$ for the S^1-equivariant line bundle which is non-equivariantly trivial and has S^1 action $\lambda \cdot (z, c) = (\lambda z, \lambda^i c)$ for $(z, c) \in S^{2n+1} \times \mathbb{C}$.

Step 1: The trivial bundle ϵ over $\mathbb{CP}^n$ pulls back to the trivial bundle with the trivial action over S^{2n+1}:

$$\begin{array}{ccc} \epsilon_0 & \longrightarrow & \epsilon \\ \downarrow & & \downarrow \\ S^{2n+1} & \xrightarrow{p} & \mathbb{CP}^n \end{array} \qquad\qquad \begin{array}{ccc} (z,c) & \longmapsto & [z], c \\ \lambda \downarrow & \nearrow & \\ (\lambda z, c) & & \end{array}$$

As above, we exhibit the S^1-action on the total space of the bundle over S^{2n+1} on the right, together with the map of total spaces.

Step 2: The bundle ℓ^{-1} pulls back to the bundle ϵ_1:

$$\begin{array}{ccc} \epsilon_1 & \longrightarrow & \ell^{-1} \\ \downarrow & & \downarrow \\ S^{2n+1} & \xrightarrow{p} & \mathbb{CP}^n \end{array} \qquad\qquad \begin{array}{ccc} (z,c) & \longmapsto & [z_0, \dots, z_n, c] \\ \lambda \downarrow & \nearrow & \\ (\lambda z, \lambda c) & & \end{array}$$

Step 3: The tangent bundle $\mathrm{T}\,\mathbb{C}^{n+1}|_{S^{2n+1}}$ is isomorphic to $(n+1)\epsilon_1$ because the action of S^1 on $\mathbb{C}^{n+1}$ is linear, and hence induces the same map on tangents. We can identify $p^*\,\mathrm{T}\,\mathbb{CP}^n$ with the sub-bundle of $(n+1)\epsilon_1$

$$\left\{(z,v) \in S^{2n+1} \times \mathbb{C}^{n+1} \mid z \cdot z = 1, v \cdot z = v \cdot iz = 0\right\}.$$

Here, the dot product is the usual Euclidean inner product on the underlying real vector space. Then $\mathrm{T}\,\mathbb{CP}^n$ can be viewed as equivalence classes of such pairs: $[z,v] = \left\{(\lambda z, \lambda v) \mid \lambda \in S^1\right\}$.

There is an evident map of bundles:

$$\begin{array}{ccc} p^*\,\mathrm{T}\,\mathbb{CP}^n & \longrightarrow & \mathrm{T}\,\mathbb{CP}^n \\ \downarrow & & \downarrow \\ S^{2n+1} & \xrightarrow{p} & \mathbb{CP}^n \end{array} \qquad\qquad \begin{array}{ccc} (z,v) & \longmapsto & [z,v] \\ \lambda \downarrow & \nearrow & \\ (\lambda z, \lambda v) & & \end{array}$$

The identification follows from the fact that the n-dimensional subbundle is a complement to

$$\mathrm{Ker}(\mathrm{T}\,p) = \left\{(z,v) \in S^{2n+1} \times \mathbb{C}^{n+1} \mid v \in \mathbb{C}z\right\}.$$

Note that we have an isomorphism

$$\begin{array}{ccc} \epsilon_0 & \longrightarrow & \mathrm{Ker}(\mathrm{T}\,p) \\ \downarrow & & \downarrow \\ S^{2n+1} & = & S^{2n+1} \end{array} \qquad\qquad \begin{array}{ccc} (z,c) & \longmapsto & (z, cz) \\ \lambda \downarrow & & \lambda \downarrow \\ (\lambda z, c) & \longmapsto & (\lambda z, \lambda cz) \end{array}$$

Step 4: Assembling the two subbundles of $\mathrm{T}\,\mathbb{C}^{n+1}|_{S^{2n+1}}$, we now have the complex analog of the classic stable trivialization of the tangent bundle of S^{2n+1} with its equivariance made clear:

$$\begin{array}{ccc} p^*\,\mathrm{T}\,\mathbb{CP}^n \oplus \epsilon_0 & \longrightarrow & (n+1)\epsilon_1 \\ \downarrow & & \downarrow \\ S^{2n+1} & = & S^{2n+1} \end{array} \qquad\qquad \begin{array}{ccc} (z,v,c) & \longmapsto & (z, v + cz) \\ \lambda \downarrow & & \lambda \downarrow \\ (\lambda z, \lambda v, c) & \longmapsto & (\lambda z, \lambda v + \lambda cz) \end{array}$$

These produce the isomorphism we require, which is the dotted map in the diagram below.

$$\begin{array}{ccc} p^* \, \mathrm{T}\, \mathbb{C}\mathbb{P}^n \oplus \epsilon_0 & \longrightarrow & \mathrm{T}\, \mathbb{C}\mathbb{P}^n \oplus \epsilon \\ \downarrow & & \downarrow \\ (n+1)\epsilon_1 & \longrightarrow & (n+1)\ell^{-1} \end{array} \qquad \begin{array}{ccc} (z, v, c) & \longmapsto & [z, v], c \\ \downarrow & & \downarrow \\ (z, v + cz) & \longmapsto & [z], ((v_i + cz_i)_0^n z) \end{array}$$

The last coordinate $((v_i + cz_i)_0^n z)$ is the $(n+1)$-tuple of multiples of z,

$$(v_0 + cz_0)z, (v_1 + cz_1)z, \ldots, (v_n + cz_n)z).$$

References

[1] John Milnor and James Stasheff, Characteristic Classes, Princeton University Press.