$\mathcal{A}(2)$ Modules and their Cohomology

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Outline

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Preview

- To learn to use the Adams spectral sequence, I strongly recommend

$$\mathrm{Ext}_{\mathcal{A}(1)}(H^*X, \mathbb{F}_2) \Longrightarrow ko_*X$$

  and

$$\mathrm{Ext}_{\mathcal{A}(1)}(\mathbb{F}_2, H^*X) \Longrightarrow ko^*X$$

  Here $A(1) = \langle Sq^1, Sq^2 \rangle$.

- Exercise: Use this method to compute ko^*RP^n for all n; H^*RP^n determines differentials, and you recover $KO^0(RP^n) = \mathbb{Z}/2^{\phi(n)}$ quite easily. Here

$$\begin{aligned}\phi(n) =& |\{k \mid 0 < k \leq n,\ k \equiv 0, 1, 2, 4 \pmod 8\}| \\ =& |\{k \mid 0 < k \leq n,\ \pi_k BO \neq 0\}|\end{aligned}$$

- $\mathcal{A}(2) = \langle Sq^1, Sq^2, Sq^4 \rangle$ is harder to work with by drawing the sort of diagrams useful for $\mathcal{A}(1)$. See Aof2
- $R = \mathbb{F}_2[w_1, g, w_2] \subset \mathrm{Ext}_{\mathcal{A}(2)}(\mathbb{F}_2, \mathbb{F}_2)$,
- $w_1 = v_1^4$, $g = Sq^0(w_1)$ and $w_2 = v_2^8$,
- As an R-module, $\mathrm{Ext}_{\mathcal{A}(2)}(\mathbb{F}_2, \mathbb{F}_2)$, is a direct sum of cyclic modules R, $R/(g)$, and $R/(g^2)$, finitely generated over $R[h_0]$.
- Free over $\mathbb{F}_2[w_1, w_2]$.
- w_1, g, and w_2 arise from the Steenrod operations
    - $Q_1 = Sq^1 Sq^2 + Sq^2 Sq^1$,
    - $P_2^1 = Sq^2 Sq^4 + Sq^4 Sq^2$, and
    - $Q_2 = Q_1 Sq^4 + Sq^4 Q_1$, respectively,
- just as h_0 and $h_1 = Sq^0(h_0)$ arise from Sq^1 and Sq^2.

$\mathrm{Ext}_{\mathcal{A}(2)}$

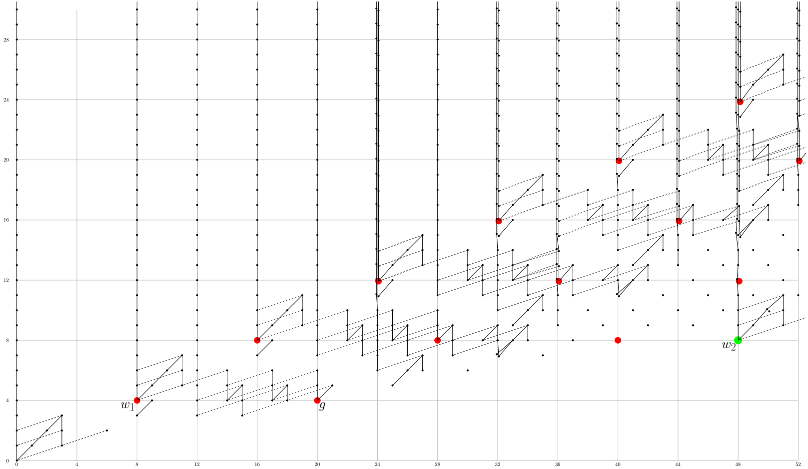


Figure 1: $\mathrm{Ext}_{A(2)}$, $0 \le n \le 52$, $0 \le s \le 30$

In the Adams spectral sequence

$$\mathrm{Ext}_{\mathcal{A}(2)}(\mathbb{F}_2, \mathbb{F}_2) \Longrightarrow \pi_* tmf$$

- E_2 is a f.g. $R[h_0]$-module and d_2 is $R_1 = \mathbb{F}_2[w_1, g, w_2^2]$ linear.
- E_3 is a f.g. $R_1[h_0]$-module and d_3 is $R_2 = \mathbb{F}_2[w_1, g, w_2^4]$ linear.
- E_4 is a f.g. $R_2[h_0]$-module and d_4 is $R_2 = \mathbb{F}_2[w_1, g, w_2^4]$ linear.
- $E_5 = E_\infty$ is a f.g. $R_2[h_0]$-module.

$E_\infty = E^0(tmf_*)$, basic block

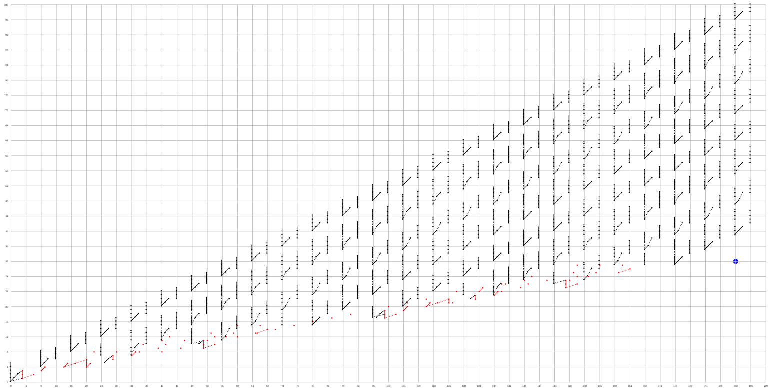


Figure 1: E_∞, $0 \le t-s < 192$. w_1-power torsion is shown in red, while black classes are free over $\mathbf{F}_2[w_1, w_2^4]$. The w_2^4 multiples are not shown.

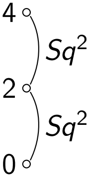
Joint with John Rognes.
p=2 throughout.

Long term goal: understand $\pi_* S$, stable homotopy of spheres, with its multiplicative structure (products, Toda brackets, cup-i products, ...).

Roughly equivalent to understanding the category of finite complexes.

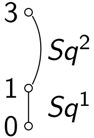
Example: Let $\eta : S^3 \longrightarrow S^2$ be the Hopf map, with cofiber $\mathbb{C}P^2$.

Then $\eta^2 \neq 0$, even stably, because, if it were zero, we could form the complex $S^0 \cup_\eta e^2 \cup_\eta e^4$ with mod 2 cohomology

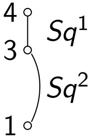


But the Adem relation $Sq^2 Sq^2 = Sq^1 Sq^2 Sq^1$ shows that no such complex exists.

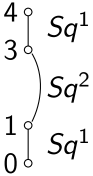
Similarly, though you can form the complexes



and



you cannot form the complex



The fact that you cannot add the top cell

$\iff \pi_2(S^0 \cup_2 e^1) = \mathbb{Z}/4$ rather than $\mathbb{Z}/2 \oplus \mathbb{Z}/2$

$\iff 0 \notin \langle 2, \eta, 2 \rangle = \{\eta^2\}$ in $\pi_* S$.

Moral: the structure of modules over (even small subalgebras of) the Steenrod algebra can tell us a lot about $\pi_* S$ and the category of finite complexes.

- Immediate goal: Study the v_2 and v_3-periodic homotopy in

$$S \longrightarrow K(tmf) \longrightarrow THH(tmf)^{tS^1}$$

- Tool: Adams spectral sequence

$$\mathrm{Ext}_{\mathcal{A}}(H^*(tmf \wedge X), \mathbb{F}_2) \Longrightarrow tmf_*(X)$$

for relevant complexes X.

The Steenrod algebra $\mathcal{A}$ and its dual

- $\mathcal{A} = H\mathbb{F}_2^* H\mathbb{F}_2$, generated by $Sq^1, Sq^2, \ldots$ modulo the Adem relations.
- Sub-Hopf algebras are most easily described in terms of the dual, which has a beautiful description in terms of automorphisms of the additive group G_a.
- $G_a(R) = (R, +) \;\cong\; \mathrm{Hom}_{\mathbb{F}_2-\mathrm{alg}}(\mathbb{F}_2[x], R)$
- Thus, $a \in \mathrm{Aut}(G_a)$ is given by $a \in \mathbb{F}_2[x]$ such that

  $$a(x+y) = a(x) + a(y), \qquad \text{so } a = \textstyle\sum_{i \geq 0} a_i x^{2^i}.$$
- Hence, if $\zeta_i : \mathrm{Aut}(G_a) \longrightarrow \mathbb{F}_2$ is $\zeta_i(a) = a_i$, then the coordinate ring of $\mathrm{Aut}(G_a)$ is

  $$\mathbb{F}_2[\zeta_1, \zeta_2, \zeta_3, \ldots]$$

Now

$$a \circ b = \sum_{i \geq 0} a_i \left(\sum_{j \geq 0} b_j x^{2^j} \right)^{2^i}$$
$$= \sum_{n \geq 0} \left(\sum_{i+j=n} a_i b_j^{2^i} \right) x^{2^n}$$

so that

$$\zeta_n(a \circ b) = \sum_{i+j=n} \zeta_i(a) \zeta_j(b)^{2^i}.$$

Coproduct inducing composition:

$$\psi(\zeta_n) = \sum_{i+j=n} \zeta_i \otimes \zeta_j^{2^i}.$$

- $\mathcal{A}_* = \mathrm{Hom}(\mathcal{A}, F_2) \;\cong\; \mathbb{F}_2[\zeta_n \mid n \geq 1]$ with this coproduct.
- More commonly written using conjugate generators $\xi_i = \chi(\zeta_i)$. The coproduct becomes

$$\psi(\xi_n) = \sum_{i+j=n} \xi_i^{2^j} \otimes \xi_j.$$

- Milnor:
  - ξ_{n+1} is dual to Q_n, where $Q_0 = Sq^1$ and $Q_n = [Sq^{2^n}, Q_{n-1}]$ if $n > 0$.
  - Sq^n is dual to ξ_1^n

Sub-Hopf algebras of $\mathcal{A}$

- Sub-Hopf algebras of $\mathcal{A}$ are dual to quotient Hopf algebras of $\mathcal{A}_*$, which have the form

$$\mathbb{F}_2[\xi_n \mid n \geq 1]/(\xi_1^{2^{r_1}}, \xi_2^{2^{r_2}}, \ldots)$$

  or equivalently

$$\mathbb{F}_2[\zeta_n \mid n \geq 1]/(\zeta_1^{2^{r_1}}, \zeta_2^{2^{r_2}}, \ldots).$$

- The *profile function* is $(r_1, r_2, \ldots)$
- It satisfies restrictions which ensure that quotient is a coalgebra.
- Profile function $(n+1, n, \ldots, 1)$ defines $\mathcal{A}(n) = \langle Sq^1, Sq^2, \ldots, Sq^{2^n} \rangle$.
- Profile function $(1, 1, \ldots, 1)$, with $n+1$ ones, defines $E(n) = E[Q_0, Q_1, \ldots, Q_n] \subset \mathcal{A}(n)$.

tmf and $\mathcal{A}(2)$

- $H^* tmf = \mathcal{A} /\!\!/ \mathcal{A}(2) \stackrel{\text{def}}{=} \mathcal{A} \otimes_{\mathcal{A}(2)} \mathbb{F}_2$,
- the Künneth theorem and untwisting isomorphism gives

$$H^*(tmf \wedge X) \cong (\mathcal{A} \otimes_{\mathcal{A}(2)} \mathbb{F}_2) \otimes H^*X \cong \mathcal{A} \otimes_{\mathcal{A}(2)} H^*X.$$

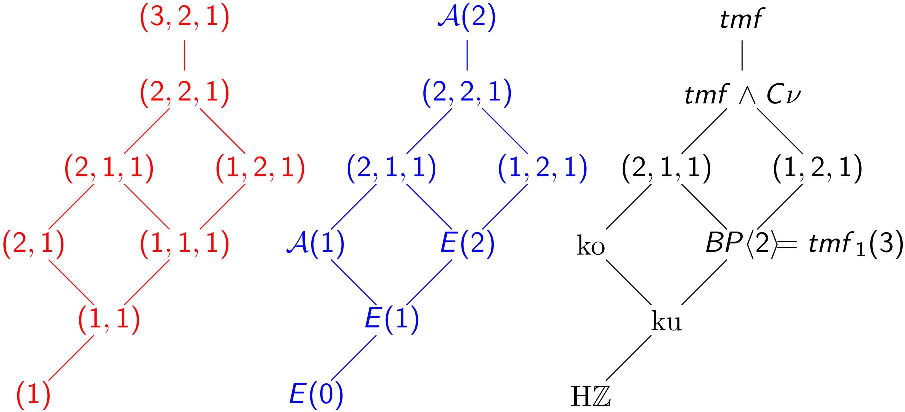
- the Adams spectral sequence

$$\mathrm{Ext}_{\mathcal{A}}(H^*(tmf \wedge X), \mathbb{F}_2) \Longrightarrow tmf_*(X)$$

then takes the simpler form

$$\mathrm{Ext}_{\mathcal{A}(2)}(H^*(X), \mathbb{F}_2) \Longrightarrow tmf_*(X)$$

Sub-Hopf algebras of $\mathcal{A}(2)$



$E(0)$

We shall work our way up to $\mathcal{A}(2)$ from its sub-Hopf algebras.

- $E(0) = E[Q_0]$ has *finite* representation type. There are only two indecomposable $E(0)$-modules:
  - the simple module $\mathbb{F}_2$, and
  - the free module $E(0)$.
- All other $E(0)$-modules are direct sums of these.

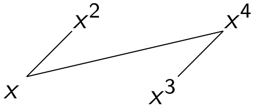
$E(1)$

$E(1) = E[Q_0, Q_1]$ and $\mathcal{A}(1) = \langle Sq^1, Sq^2 \rangle$ have *tame* representation type: their bounded below indecomposable modules of finite-type fall into a small number of families of simply parameterized modules.

- Over $E(1)$, they are $E(1)$ and the lightning flashes, parameterized by substrings of the bi-infinite string

$$\cdots Q_1 Q_0^{-1} Q_1 Q_0^{-1} \cdots$$

- For example, $H^* RP^4$ corresponds to the string $Q_0^{-1} Q_1 Q_0^{-1}$:



$\mathcal{A}(1)$

- Over $\mathcal{A}(1)$ the classification has been carried out for ungraded modules over $GF(4)$ by William Crawley-Boevey.
- To describe the modules topologists care about:
  - Discard those which cannot be graded appropriately.
  - Do the Galois descent from $GF(4)$ to $GF(2)$.
- Crawley-Boevey's classification uses *admissable* words in the alphabet

$$\{a_2, a_1, a_0, a_{-1}, a_{-2}\} \cup \{b_1, b_{-1}\},$$

where $c_{-i} = c_i^{-1}$ and
  - $b_1 = Sq^1$,
  - $a_1 = Sq^2$,
  - $a_2 = Sq^2 Sq^1 Sq^2$, and
  - $a_0 = Sq^2 Sq^2 = Sq^1 Sq^2 Sq^1$.

$E(2)$ and larger algebras

- Unfortunately, $E(2)$ has *wild* representation type: classifying even the finite modules over it would require solving the (unsolvable) word problem.
- This then also applies to all the algebras containing $E(2)$.
- However, we are not interested in all modules, but only in certain special ones. For these, some progress may be possible. For example, see Benson, "Representations of Elementary Abelian p-Groups and Vector Bundles".

The stable module category

A finite dimensional cocommutative Hopf algebra B is a Frobenius algebra, so that

$$\text{projective} \iff \text{free} \iff \text{injective.}$$

We then define the *stable module category* St(B-Mod):

- objects are B-modules,
- morphisms are *stable equivalence classes* of homomorphisms

$$[M, N] = \mathrm{Hom}_B(M, N)/\sim$$

where $f \sim g$ iff $f - g$ factors through a free module.

Then

- Every B-module can be written (non-uniquely) as $F \oplus \overline{M}$, where
  - F is free
  - $\overline{M}$ is *reduced* (has no free sub-modules or quotient modules).
- M is stably isomorphic to N iff $\overline{M}$ is isomorphic to $\overline{N}$.
- For $s > 0$, $\mathrm{Ext}_B^s(M, N)$ depends only on $\overline{M}$ and $\overline{N}$
- St(B-Mod) is tensor triangulated with a triangle

$$\Omega Z \longrightarrow X \longrightarrow Y \longrightarrow Z$$

for each exact sequence

$$0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$$

Here, ΩZ is stably the kernel of an epimorphism $\mathrm{Free} \longrightarrow Z$.

Margolis homology

$P_t^s = (\xi_t^{2^s})^*$ for $s < t$ satisfies $(P_t^s)^2 = 0$.
$(P_t^0 = Q_{t-1})$
Define the *Margolis homology*

$$H(-, P_t^s) = \ker(P_t^s)/\mathrm{im}(P_t^s).$$

For a sub-Hopf algebra B of the Steenrod algebra,

Theorem (Adams-Margolis)

If M and N are bounded below B-modules of finite type, then

- $M \xrightarrow{f} N$ *is a stable isomorphism* $\iff$
- $H(f, P_t^s)$ *is an isomorphism for each* $P_t^s \in B$.

The P^s_t's in B are *totally ordered* by degree.

Let I_B be the (totally ordered) set of $P^s_t \in B$.

Definition

For any subset $J \subset I_B$, we say that $M \in B{-}\mathrm{Mod}$ is *J-local* if

$$H(M, P^s_t) = 0$$

for $P^s_t \in I_B \setminus J$.

Theorem (Margolis)

Suppose $I_B = J_0 \coprod J_1$, with J_0 an initial seqment and J_1 a final segment of I_B. For a bounded-below B-module of finite type, there exists a unique triangle

$$L_{J_1}(M) \xleftarrow{\epsilon} M \xleftarrow{\iota} L_{J_0}(M)$$

in which

- $L_{J_i}(M)$ *is J_i-local, $i = 0, 1$*
- ϵ *is an isomorphism in P^s_t homology for $P^s_t \in J_1$*
- ι *is an isomorphism in P^s_t homology for $P^s_t \in J_0$*
- $L_{J_i}M$ *is bounded-below of finite type.*

- The case $M = \mathbb{F}_2$ is sufficient since the L_J are smashing.
- The functors L_{J_i} are called *Margolis localizations*, though L_{J_0} is really a colocalization.
- By composition, we can define $L_J(M)$ for any subinterval J of I_B.
- For $B = E(1)$ or $\mathcal{A}(1)$, $I_B = \{Q_0 < Q_1\}$ so this reduces to a single exact triangle

$$L_1(M) \xleftarrow{\epsilon} M \xleftarrow{\iota} L_0(M)$$

in which $L_i(M)$ is Q_i-local.

v_0 periodicity

Let B be any sub-Hopf algebra of $\mathcal{A}$ containing Q_0.

Theorem

If M is a Q_0-local B-module, then $\Omega M \simeq \Sigma M$.

Proof.

Tensor M with $0 \longleftarrow \mathbb{F}_2 \longleftarrow E(0) \longleftarrow \Sigma\mathbb{F}_2 \longleftarrow 0$, where $E(0)$ has its unique B-module structure. Since M has only Q_0 homology and $E(0)$ has no Q_0 homology, $M \otimes E(0)$ is free. □

Remark

If M is Q_0-local and reduced, $\mathrm{Ext}_B(M, \mathbb{F}_2) \cong \mathbb{F}_2[h_0] \otimes \mathrm{Ext}_B^0(M, \mathbb{F}_2)$.

v_1 periodicity

Theorem

If M is a Q_1-local $E(1)$-module then $\Omega M \simeq \Sigma^3 M$.

Proof.

The same argument works, using

$$0 \longleftarrow \mathbb{F}_2 \longleftarrow E[Q_1] \longleftarrow \Sigma^3\mathbb{F}_2 \longleftarrow 0.$$

□

Remark

If M is Q_1-local and reduced then

- $\mathrm{Ext}_{E(1)}(M, \mathbb{F}_2) \;\cong\; \mathbb{F}_2[v_1] \otimes \mathrm{Ext}^0_{E(1)}(M, \mathbb{F}_2)$.

v_1^4 periodicity

Theorem

If M is a Q_1-local $\mathcal{A}(1)$-module then $\Omega^4 M \simeq \Sigma^{12} M$.

Proof.

Use the exact sequence

$$0 \longleftarrow \mathbb{F}_2 \longleftarrow \mathcal{A}(1)/\!\!/\mathcal{A}(0) \longleftarrow \Sigma^2\mathcal{A}(1) \longleftarrow$$

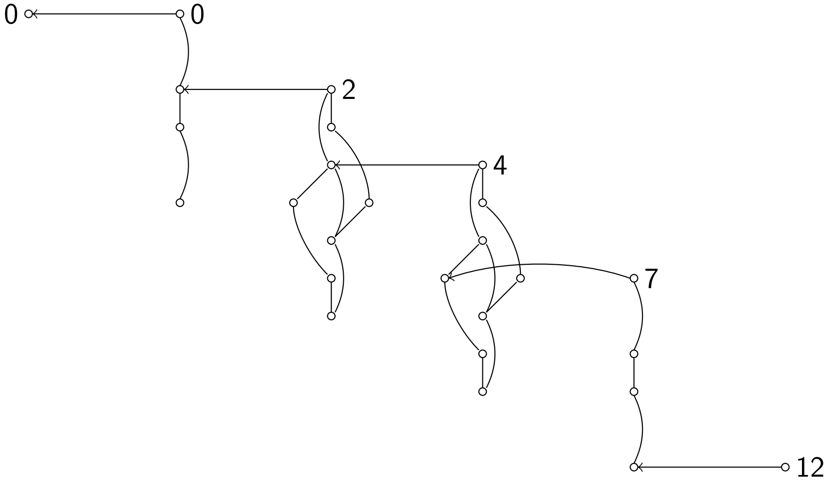
$$\Sigma^4\mathcal{A}(1) \longleftarrow \Sigma^7\mathcal{A}(1)/\!\!/\mathcal{A}(0) \longleftarrow \Sigma^{12}\mathbb{F}_2 \longleftarrow 0$$

as above. Note that $\mathcal{A}(1)$ and $\mathcal{A}(1)/\!\!/\mathcal{A}(0)$ are Q_1-acyclic $\mathcal{A}(1)$-modules, so that they become free after tensoring them with a Q_1-local module. □

Remark

This arises from the first four stages of the Postnikov tower of ko.

v_1^4 periodicity

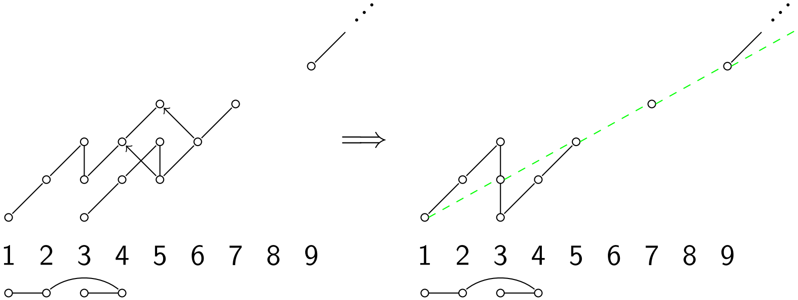


For general M, the sequence tensored with M and spliced gives a 4-periodic complex from which we get a spectral sequence converging to $\mathrm{Ext}_{\mathcal{A}(1)}(M, \mathbb{F}_2)$.
At E_1,

- the terms $s \equiv 1, 2 \pmod 4$ are suspensions of M concentrated in homological degree s, while
- the terms $s \equiv 0, 3 \pmod 4$ are $\mathrm{Ext}^*_{\mathcal{A}(0)}(M, \mathbb{F}_2)$, in homological degrees $\geq s$.

For example, consider $M = H^* RP^4$: (Consider also $M = \mathbb{F}_2$?)

$ko_* RP^4$



Alternatively, since

- any M sits in a triangle

$$L_1 M \longleftarrow M \longleftarrow L_0 M$$

- $\Omega L_0 M \simeq \Sigma L_0 M$
- $\Omega^4 L_1 M \simeq \Sigma^{12} L_0 M$

we need only compute $\mathrm{Ext}^0_{\mathcal{A}(1)}(L_0 M, \mathbb{F}_2)$ and $\mathrm{Ext}^i_{\mathcal{A}(1)}(L_1 M, \mathbb{F}_2)$ for $0 \leq i \leq 3$.

These give $\mathrm{Ext}_{\mathcal{A}(1)}(L_0 M, \mathbb{F}_2)$ and $\mathrm{Ext}_{\mathcal{A}(1)}(L_1 M, \mathbb{F}_2)$, and we can then compute $\mathrm{Ext}_{\mathcal{A}(1)}(M, \mathbb{F}_2)$ from the long exact sequnce.

v_2 periodicity

Theorem

If M is a Q_2 local $E(2)$-module then $\Omega M \simeq \Sigma^7 M$.

Proof.

This is a general fact. If $A = B \otimes E[Q]$ and M is an A-module which is free over B, then tensoring M with

$$0 \longleftarrow \mathbb{F}_2 \longleftarrow E[Q] \longleftarrow \Sigma^{|Q|}\mathbb{F}_2 \longleftarrow 0$$

shows that $\Omega M \simeq \Sigma^{|Q|} M$. □

Remark

In this context, it is easy to construct a Q-localization $M \longrightarrow L_Q M$, a Q-colocalization $C_Q \longrightarrow M$, and a Tate module $T_Q M$ which sits in a triangle $C_Q \longrightarrow T_Q M \longrightarrow L_Q M$.

$\mathcal{A}(2)$

In $\mathcal{A}(2)$ the homologies which determine stable isomorphism are

$$I_{\mathcal{A}(2)} = [q_0, q_1, q_2, q_3] = [Q_0 < Q_1 < P_2^1 < Q_2].$$

The Margolis localizations and colocalizations they determine are

$$\begin{array}{ccccccccc}
M & = & M[0\text{–}3] & \leftarrow & M[0\text{–}2] & \leftarrow & M[0\text{–}1] & \leftarrow & M[0] \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & M[1\text{–}3] & \leftarrow & M[1\text{–}2] & \leftarrow & M[1] & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & M[2\text{–}3] & \leftarrow & M[2] & & & & \\
 & & \downarrow & & & & & & \\
 & & M[3] & & & & & &
\end{array}$$

v_2^8 periodicity

Theorem

If M is a Q_2-local $\mathcal{A}(2)$-module then $\Omega^8 M \simeq \Sigma^{56} M$.

Proof.

There is an exact complex of $\mathcal{A}(2)$-modules

$$0 \longleftarrow \mathbb{F}_2 \longleftarrow P_0 \longleftarrow P_1 \longleftarrow P_2 \longleftarrow P_3$$

$$\longleftarrow P_4 \longleftarrow P_5 \longleftarrow P_6 \longleftarrow P_7 \longleftarrow \Sigma^{56}\mathbb{F}_2 \longleftarrow 0$$

in which each of the P_i is Q_2-acyclic. □

Let us write $\mathcal{A}(2)$ as $\mathcal{A}$ for brevity.

- $P_0 = \mathcal{A}/(Sq^1, Sq^2)$ so that
$$\mathrm{Ext}_{\mathcal{A}}(P_0, \mathbb{F}_2) = \mathrm{Ext}_{\mathcal{A}(1)}(\mathbb{F}_2, \mathbb{F}_2) \Rightarrow \mathrm{ko}_*$$
- $P_1 = \Sigma^4 \mathcal{A}/(Sq^1, Sq^2 Sq^3)$ giving
$$\mathrm{Ext}_{\mathcal{A}}(P_1, \mathbb{F}_2) = \mathrm{Ext}_{\mathcal{A}(1)}(\Sigma^4 \mathbb{F}_2, \mathbb{F}_2) \Rightarrow \Sigma^4 \mathrm{ksp}_*$$
- $P_2 = \Sigma^8 \mathcal{A}/(Sq^1)$ giving
$$\mathrm{Ext}_{\mathcal{A}}(P_2, \mathbb{F}_2) = \mathrm{Ext}_{\mathcal{A}(0)}(\Sigma^8 \mathbb{F}_2, \mathbb{F}_2) \Rightarrow \Sigma^8 \mathrm{H}\mathbb{Z}_*$$
- $P_3 = \Sigma^{15} \mathcal{A} \oplus \Sigma^{18} \mathcal{A}/(Sq^1, 0), (Sq^3, 0), (Sq^4, Sq^1), (Sq^4 Sq^2, Sq^3)$ so that
$$0 \longleftarrow \Sigma^{18} \mathcal{A} /\!\!/ E(1) \longleftarrow P_3 \longleftarrow \Sigma^{15} \mathcal{A} /\!\!/ E(1) \longleftarrow 0$$
Thus, $\mathrm{Ext}_{\mathcal{A}}(P_3, \mathbb{F}_2)$ can be calculated as an extension of two copies of the Adams spectral sequence converging to ku_*.

- $P_4 = (\Sigma^{22}\mathcal{A} \oplus \Sigma^{24}\mathcal{A}) \,/\, \langle (Sq^1, 0), (Q_1, Sq^1), (0, Q_1) \rangle$, giving the first Adams cover of ku.
- $P_5 = \Sigma^{26}\mathcal{A}/(Sq^1) \oplus \Sigma^{24}\mathcal{A}/(Sq^2)$, giving an $\mathrm{H}\mathbb{Z}$ and the first Postnikov cover of ko.
- $P_6 = \Sigma^{33}\mathcal{A}/(Sq^1, Q_1) \oplus \Sigma^{36}\mathcal{A}/(Sq^{(0,2)})$, giving a ku and the bottom edge of a wedge: Ext of the second summand is $\mathbb{F}_2[\sqrt[4]{g}]$, polynomial on a class in $(t-s, s) = (5, 1)$ whose fourth power is g.
- $P_7 = (\Sigma^{39}\mathcal{A} \oplus \Sigma^{39}\mathcal{A}) \,/\, \langle (Sq^1, Sq^1), (Q_1, 0), (Sq^{(0,2)}, 0), (0, Sq^2) \rangle$,

$$0 \longleftarrow \Sigma^{39}\mathcal{A}/\!\!/\mathcal{A}(1) \longleftarrow P_7 \longleftarrow \Sigma^{39}\mathcal{A}/\!\!/E[Sq^{(0,1)}, Sq^{(0,2)}] \longleftarrow 0$$

Thus, $\mathrm{Ext}_{\mathcal{A}}(P_7, \mathbb{F}_2)$ can be calculated as an extension of the Adams spectral sequence converging to ko_* and the ‘wedge’, which has $\mathrm{Ext} = \mathbb{F}_2[v_1, \sqrt[4]{g}]$.

Advertisement - FPMods

- A sage package for doing calculations with finitely presented $\mathcal{A}$-modules.
- Written by Mike Catanzaro as his master's thesis.
- Allowed to become moribund, as new versions of sage broke some things.
- Recently resuscitated by Sverre Lunøe-Nielsen.
- Here is the sage code that verifies my claim about P_7:

```
sage: A2 = SteenrodAlgebra(p=2,profile=(3,2,1))
sage: load('fpmods.py')
sage: P7 = FP_Module([0,0],
            [[Sq(1),Sq(1)],[Sq(0,1),0],[Sq(0,2),0],[0,Sq(2)]]
     algebra=A2)
sage: ko = FP_Module([0],[[Sq(1)],[Sq(2)]],algebra=A2)
sage: p = FP_Hom(P7,ko,[[0],[1]])
sage: K,j = p.kernel()
sage: K.degs
[0]
sage: K.rels
[[Sq(0,1)], [Sq(0,2)]]
```

Calculating $\text{Ext}_{\mathcal{A}(2)}$

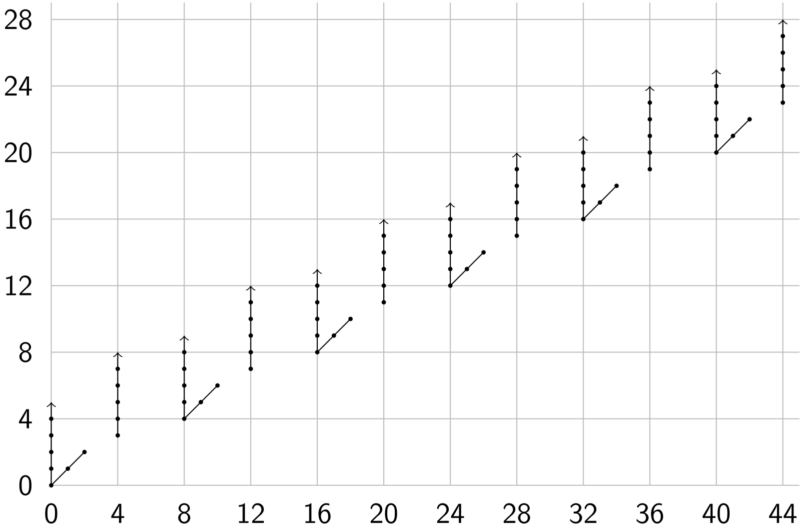


Figure: P_0

Calculating $\text{Ext}_{\mathcal{A}(2)}$

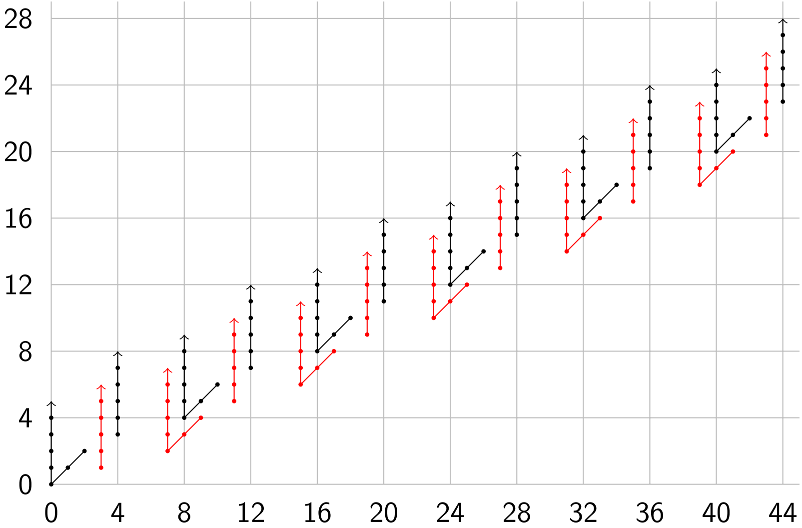


Figure: P_0, P_1

Calculating $\text{Ext}_{\mathcal{A}(2)}$

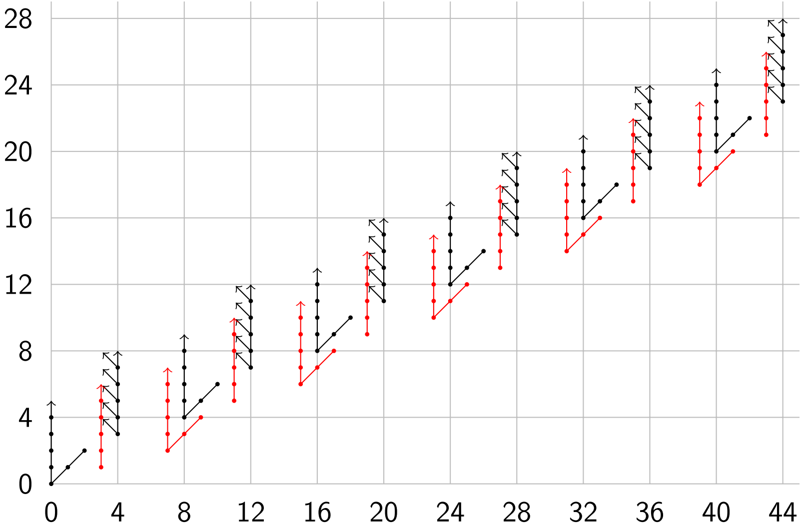


Figure: $P_0 \xrightarrow{d_1} P_1$

Calculating $\mathrm{Ext}_{\mathcal{A}(2)}$

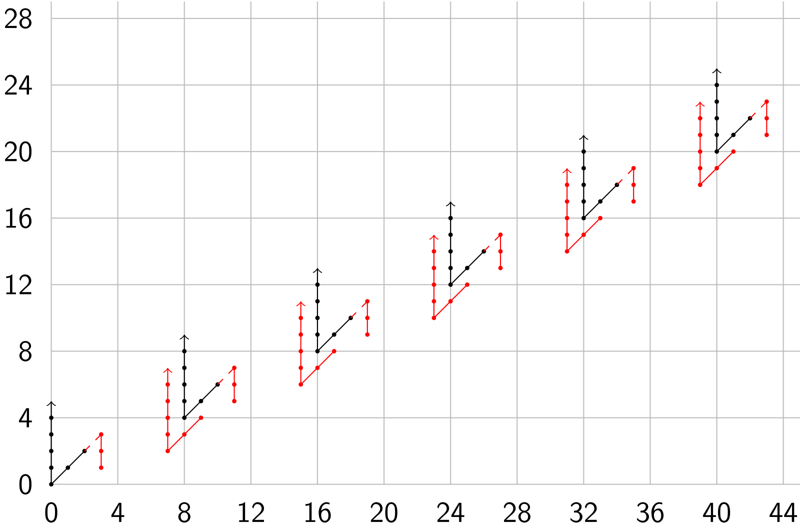


Figure: $H(P_0 \xrightarrow{d_1} P_1)$

Calculating $\mathrm{Ext}_{\mathcal{A}(2)}$

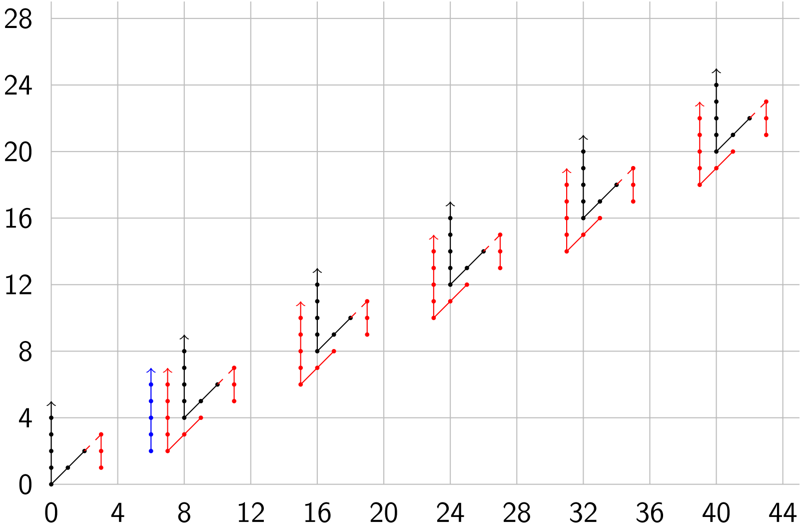


Figure: $H(P_0 \xrightarrow{d_1} P_1)$, P_2

Calculating $\mathrm{Ext}_{\mathcal{A}(2)}$

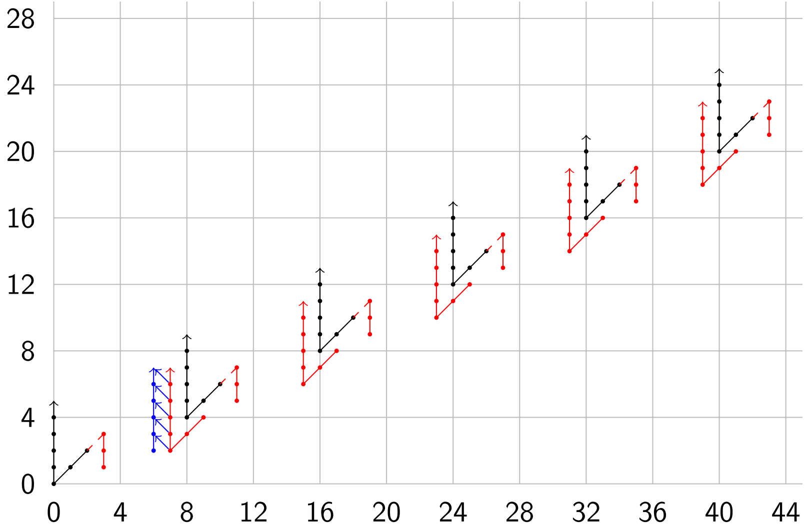


Figure: $H(P_0 \xrightarrow{d_1} P_1) \xrightarrow{d_1} P_2$

Calculating $\text{Ext}_{\mathcal{A}(2)}$

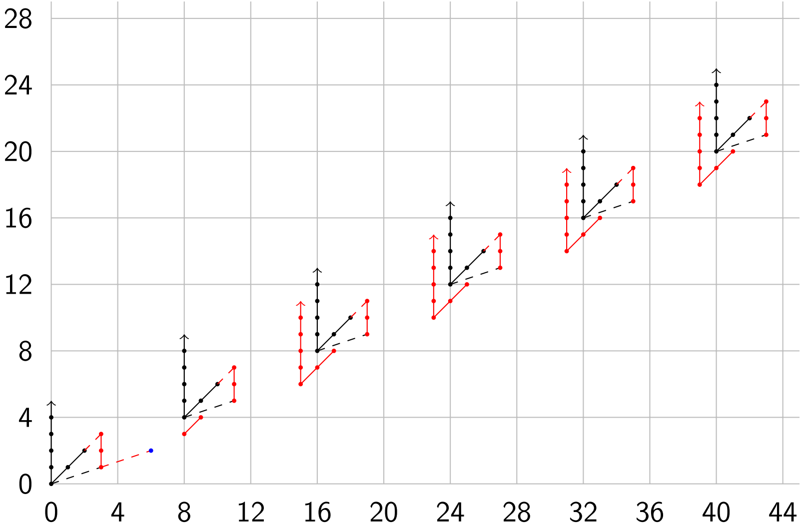


Figure: $H(P_0 \xrightarrow{d_1} P_1 \xrightarrow{d_1} P_2)$

Calculating $\mathrm{Ext}_{\mathcal{A}(2)}$

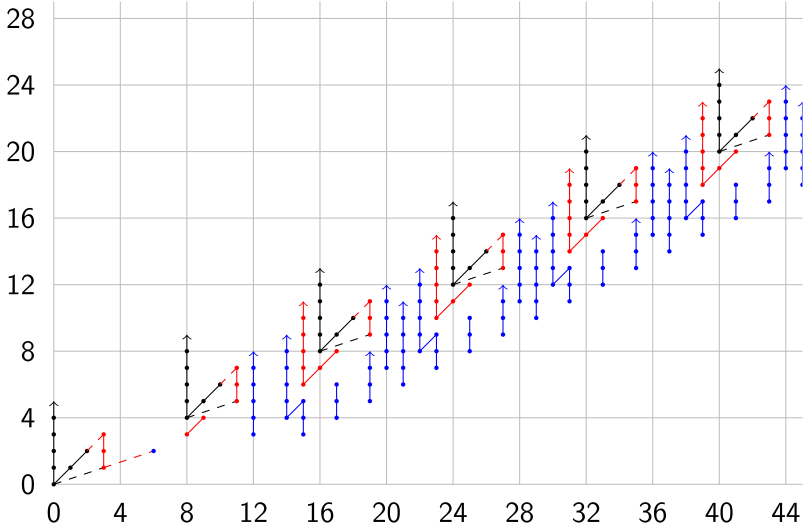


Figure: $H(P_0 \xrightarrow{d_1} P_1 \xrightarrow{d_1} P_2)$, P_3

Calculating $\mathrm{Ext}_{\mathcal{A}(2)}$

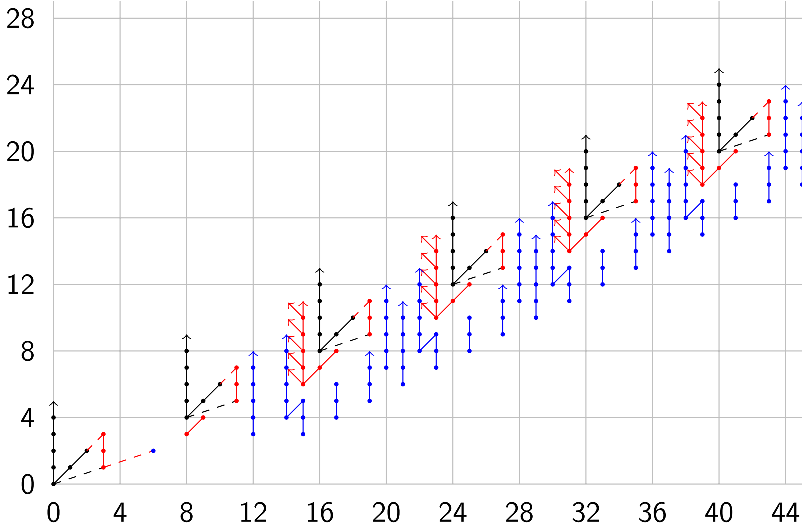


Figure: $H(P_0 \xrightarrow{d_1} P_1 \xrightarrow{d_1} P_2) \xrightarrow{d_2} P_3$

Calculating $\mathrm{Ext}_{\mathcal{A}(2)}$

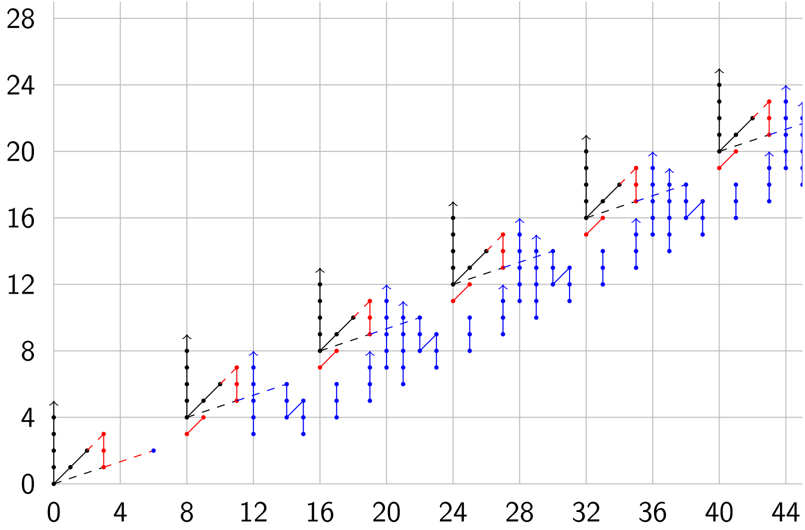


Figure: $H(P_0 \xrightarrow{d_1} P_1 \xrightarrow{d_1} P_2 \xrightarrow{d_2} P_3)$

Calculating $\mathrm{Ext}_{\mathcal{A}(2)}$

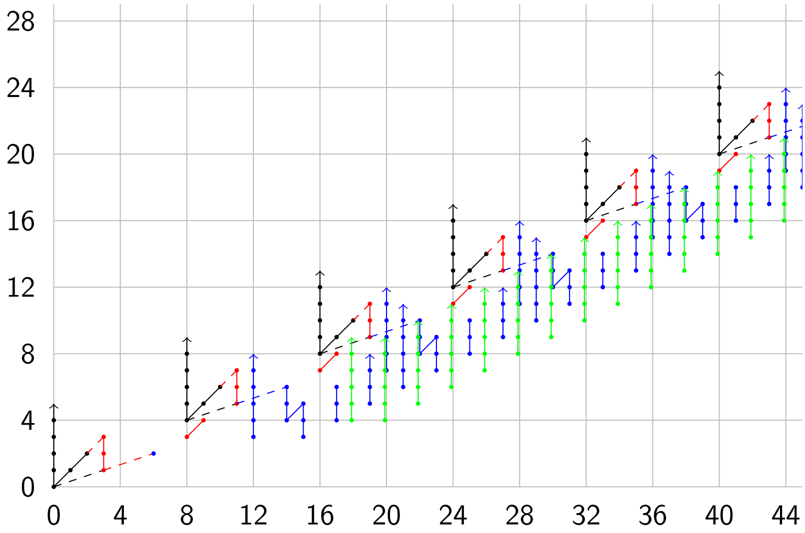


Figure: $H(P_0 \xrightarrow{d_1} P_1 \xrightarrow{d_1} P_2 \xrightarrow{d_2} P_3)$, P_4

$\mathrm{Ext}_{\mathcal{A}(2)}$

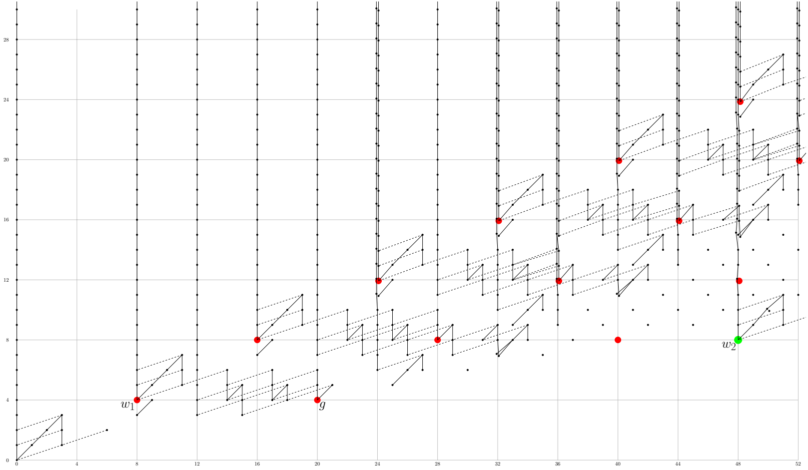


Figure 1: $\mathrm{Ext}_{\mathcal{A}(2)}$, $0 \le n \le 52$, $0 \le s \le 30$

Similar calculations

- There is a v_2^4 sequence for $H^*(C(\nu))$ giving a simple calculation of $\mathrm{Ext}_{\mathcal{A}(2)}(H^* C\nu, \mathbb{F}_2)$.
- v_2^8 sequences with small, well understood terms, exist for the cofibers of 2 and η, as well as other complexes useful in computing $THH_*(tmf)$.

$\mathrm{Ext}_{\mathcal{A}(2)}$ again

Theorem (Iwai and Shimada)

The cohomology of $A(2)$ is

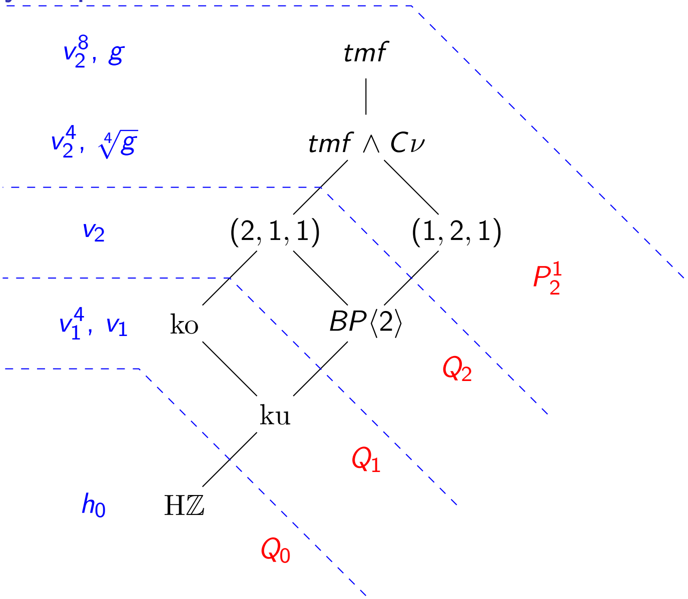
$$\mathrm{Ext}_{A(2)}(\mathbb{F}_2, \mathbb{F}_2) = \mathbb{F}_2[h_0, h_1, h_2, c_0, d_0, e_0, g, \alpha, \beta, \gamma, \delta, w_1, w_2]/I.$$

The ideal I has 56 generators:

- $h_0 h_1,\ h_1 h_2,\ h_0^2 h_2 - h_1^3,\ h_0 h_2^2,\ h_2^3$
- $\ldots$
- $c_0\gamma - h_1\delta,\ \beta\gamma - g^2,\ d_0^2 - gw_1,\ \gamma\delta - h_1 c_0 w_2,$
- $\gamma^2 - h_1^2 w_2 - g\beta^2,\ \alpha^4 - h_0^4 w_2 - w_1 g^2$

- Free over $\mathbb{F}_2[w_1, w_2]$; here w_1 and w_2 restrict to v_1^4 and v_2^8, resp.
- A sum of cyclic $R = \mathbb{F}_2[g, w_1, w_2]$-modules isomorphic to R, $R/(g)$ and $R/(g^2)$.
- Four infinite families, $h_0^i \alpha^j$, $i \geq 0$, $0 \leq j \leq 3$.
- Thirty-two other summands.
- E_3, E_4 and $E_5 = E_\infty$ are then modules over $R_1 = \mathbb{F}_2[g, w_1, w_2^2]$ and $R_2 = \mathbb{F}_2[g, w_1, w_2^4]$ resp. Mostly cyclic.

Summary of periodicities



E_∞

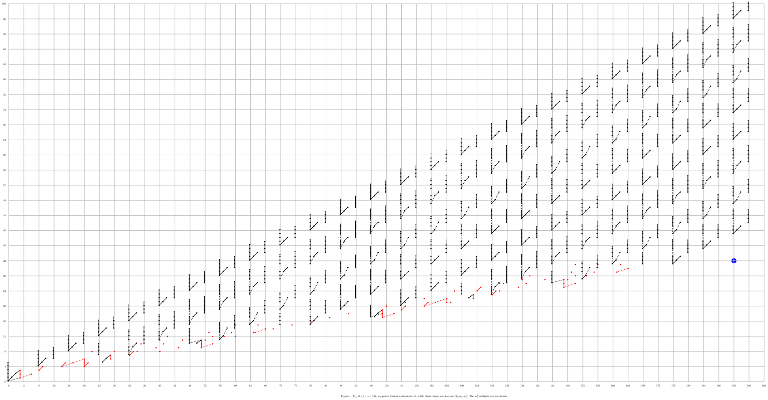


Figure: E_∞, $0 \leq t - s < 192$. w_1-power torsion is shown in red, while black classes are free over $\mathbf{F}_2[w_1, w_2^4]$. The w_2^4 multiples are not shown.

Muchas gracias