$\mathcal{A}(2)$ Modules and their Cohomology

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Outline

- Joint work with John Rognes on $THH(tmf)$ and its circle action.
- Goal: Study the v_3-periodic homotopy in

$$S \longrightarrow K(tmf) \longrightarrow THH(tmf)^{tS^1}$$

- $p = 2$ and $H = H\mathbf{F}_2$ in this talk.
- Tool: Adams spectral sequence

$$\mathrm{Ext}_{\mathcal{A}}(H^*(tmf \wedge X), \mathbf{F}_2) \Longrightarrow tmf_*(X)$$

for relevant compexes X.

tmf and $\mathcal{A}(2)$

- Since $H^* tmf = \mathcal{A}/\!/\mathcal{A}(2) = \mathcal{A} \otimes_{\mathcal{A}(2)} \mathbf{F}_2$, the Adams spectral sequence

$$\mathrm{Ext}_{\mathcal{A}}(H^*(tmf \wedge X), \mathbf{F}_2) \Longrightarrow tmf_*(X)$$

  takes the simpler form

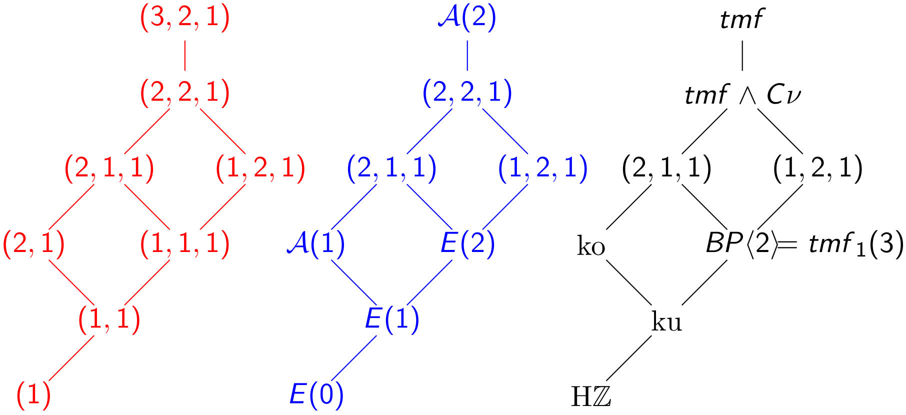
$$\mathrm{Ext}_{\mathcal{A}(2)}(H^*(X), \mathbf{F}_2) \Longrightarrow tmf_*(X)$$

- Sub-Hopf algebras of the Steenrod algebra like $\mathcal{A}(2)$ are classified by their profile functions:
  let $(r_1, r_2, \ldots, r_k)$ denote the sub-Hopf algebra of $\mathcal{A}$ dual to

$$\mathbf{F}_2[\overline{\xi}_1, \ldots, \overline{\xi}_k]/(\overline{\xi}_1^{2^{r_1}}, \ldots, \overline{\xi}_k^{2^{r_k}}).$$

- $\mathcal{A}(2) = (3, 2, 1)$ is generated by Sq^1, Sq^2, Sq^4.

Sub-Hopf algebras of $\mathcal{A}(2)$

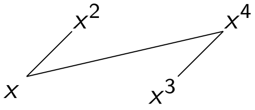


$E(0)$

- $E(0) = E[Q_0]$ has *finite* representation type. There are only two indecomposable $E(0)$-modules:
  - the simple module $\mathbf{F}_2$, and
  - the free module $E(0)$.
- All other $E(0)$-modules are direct sums of these.

$E(1)$

- $E(1) = E[Q_0, Q_1]$ and $\mathcal{A}(1) = \langle Sq^1, Sq^2 \rangle$ have *tame* representation type: their indecomposable modules of finite-type fall into a small number of families of simply parameterized modules.
- Over $E(1)$, the indecomposable modules of finite type are $E(1)$ and the lightning flashes, parameterized by substrings of the bi-infinite string
$$\cdots Q_1 Q_0^{-1} Q_1 Q_0^{-1} \cdots$$
- For example, $H^* RP^4$ corresponds to the string $Q_0^{-1} Q_1 Q_0^{-1}$:



$\mathcal{A}(1)$

- Over $\mathcal{A}(1)$ the classification has been carried out for ungraded modules over $GF(4)$ by William Crawley-Boevey.
- To describe the modules topologists care about:
  - Discard those which cannot be graded appropriately.
  - Do the Galois descent from $GF(4)$ to $GF(2)$.
- Crawley-Boevey's classification uses *admissable* words in the alphabet

$$\{a_2, a_1, a_0, a_{-1}, a_{-2}\} \cup \{b_1, b_{-1}\},$$

where $c_{-i} = c_i^{-1}$ and
  - $b_1 = Sq^1$,
  - $a_1 = Sq^2$,
  - $a_2 = Sq^2 Sq^1 Sq^2$, and
  - $a_0 = Sq^2 Sq^2 = Sq^1 Sq^2 Sq^1$.

$E(2)$ and larger algebras

- Unfortunately, $E(2)$ has *wild* representation type: classifying even the finite modules over it would require solving the (unsolvable) word problem.
- This then also applies to all the algebras containing $E(2)$.
- However, we are not interested in all modules, but only in certain special ones. For these, some progress may be possible. For example, see Benson, "Representations of Elementary Abelian p-Groups and Vector Bundles".

A finite dimensional cocommutative Hopf algebra B is a Frobenius algebra, so that

$$\text{projective} \iff \text{free} \iff \text{injective.}$$

We then define the *stable module category* St(B-Mod): objects are B-modules, and morphisms are *stable equivalence classes* of homomorphisms

$$[M, N] = \mathrm{Hom}_B(M, N)/\sim$$

where $f \sim g$ iff $f - g$ factors through a free module. Then

- Every B-module can be written (non-uniquely) as $F \oplus \overline{M}$, where F is free and $\overline{M}$ is *reduced* (has no free sub-modules or quotient modules).
- M is stably isomorphic to N iff $\overline{M}$ is isomorphic to $\overline{N}$.
- For $s > 0$, $\mathrm{Ext}^s_B(M, N)$ depends only on $\overline{M}$ and $\overline{N}$
- St(B-Mod) is tensor triangulated with a triangle $\Omega Z \longrightarrow X \longrightarrow Y \longrightarrow Z$ for each exact sequence

$$0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$$

Margolis homology

In the Steenrod algebra we have elements $P^s_t = (\xi_t^{2^s})^*$ for $s < t$ which satisfy $(P^s_t)^2 = 0$. We may then define the *Margolis homology*

$$H(-, P^s_t) = \frac{\ker(P^s_t)}{\operatorname{im}(P^s_t)}$$

For a sub-Hopf algebra B of the Steenrod algebra we have

Theorem (Adams-Margolis)

In B-Mod, $f : M \longrightarrow N$ is a stable isomorphism $\iff$ $H(f, P^s_t)$ is an isomorphism for each $P^s_t \in B$.

The P^s_t's in B are totally ordered by degree. Let I_B be the (ordered) set of $P^s_t \in B$.

Definition

For any subset $J \subset I_B$, we say that $M \in B-$Mod is *J-local* if

$$H(M, P^s_t) = 0$$

Theorem (Margolis)

If $I_B = J_0 \coprod J_1$, with J_0 an initial seqment and J_1 a final segment of I_B, then there exists a unique triangle

$$L_{J_1}(M) \xleftarrow{\epsilon} M \xleftarrow{\iota} L_{J_0}(M)$$

in which

- *$L_{J_i}(M)$ is J_i-local, $i = 0, 1$*
- *ϵ is an isomorphism in P^s_t homology for $P^s_t \in J_1$*
- *ι is an isomorphism in P^s_t homology for $P^s_t \in J_0$*

- The functors L_{J_i} are called *Margolis localizations*, though L_{J_0} is really a colocalization.
- By composition, we can define $L_J(M)$ for any subinterval J of I_B.
- For $E(1)$ or $\mathcal{A}(1)$ modules, this reduces to a single exact triangle

$$L_1(M) \xleftarrow{\epsilon} M \xleftarrow{\iota} L_0(M)$$

in which $L_i(M)$ is Q_i-local.

v_0 periodicity

Let B be any sub-Hopf algebra of $\mathcal{A}$ containing Q_0.

Theorem

If M is a Q_0-local B-module, then $\Omega M \simeq \Sigma M$.

Proof.

Tensor M with the short exact sequence

$$0 \longleftarrow \mathbf{F}_2 \longleftarrow E(0) \longleftarrow \Sigma\mathbf{F}_2 \longleftarrow 0$$

where $E(0)$ has its unique B-module structure. Since M has only Q_0 homology and $E(0)$ has no Q_0 homology, $M \otimes E(0)$ is free. □

Remark

If M is such a module, it follows that

- $\mathrm{Ext}_B(M, \mathbf{F}_2) \;\cong\; \mathbf{F}_2[h_0] \otimes \mathrm{Ext}^0_B(M, \mathbf{F}_2)$.

v_1 periodicity

Theorem

If M is a Q_1-local $E(1)$-module then $\Omega M \simeq \Sigma^3 M$.

Proof.

The same argument works, using

$$0 \longleftarrow \mathbf{F}_2 \longleftarrow E[Q_1] \longleftarrow \Sigma^3 \mathbf{F}_2 \longleftarrow 0.$$

□

Remark

If M is Q_1-local and reduced then

- $\mathrm{Ext}_{E(1)}(M, \mathbf{F}_2) \ \cong \ \mathbf{F}_2[v_1] \otimes \mathrm{Ext}^0$.

v_1^4 periodicity

Theorem

If M is a Q_1-local $\mathcal{A}(1)$-module then $\Omega^4 M \simeq \Sigma^{12} M$.

Proof.

Use the exact sequence

$$0 \longleftarrow \mathbf{F}_2 \longleftarrow \mathcal{A}(1)/\!/\mathcal{A}(0) \longleftarrow \Sigma^2\mathcal{A}(1) \longleftarrow \Sigma^4\mathcal{A}(1)$$

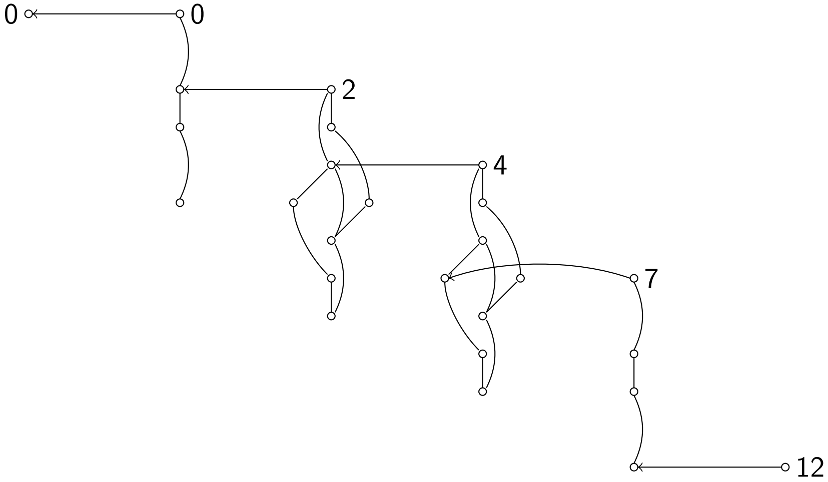
$$\longleftarrow \Sigma^7\mathcal{A}(1)/\!/\mathcal{A}(0) \longleftarrow \Sigma^{12}\mathbf{F}_2 \longleftarrow 0$$

as above, noting that both $\mathcal{A}(1)$ and $\mathcal{A}(1)/\!/\mathcal{A}(0)$ are Q_1-acyclic $\mathcal{A}(1)$-modules, so that they become free after tensoring them with a Q_1-local module. □

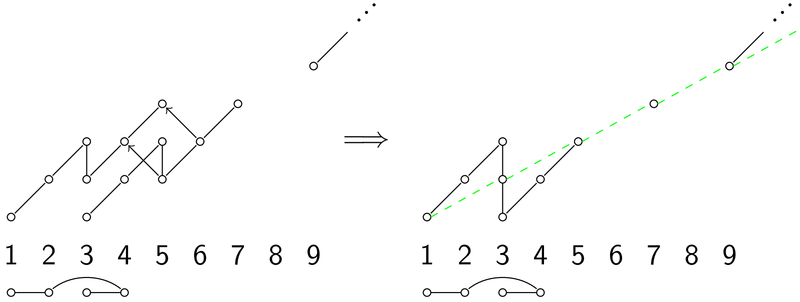
Remark

This arises from the first four stages of the Postnikov tower of ko.

v_1^4 periodicity



$ko_* RP^4$



For general M, the sequence tensored with M and spliced gives a 4-periodic complex from which we get a spectral sequence converging to $\mathrm{Ext}_{\mathcal{A}(1)}(M, \mathbf{F}_2)$.
At E_1,

- the terms $s \equiv 1, 2 \pmod 4$ are suspensions of M concentrated in homological degree s, while
- the terms $s \equiv 0, 3 \pmod 4$ are $\mathrm{Ext}^{s_1}_{\mathcal{A}(0)}(M, \mathbf{F}_2)$, in homological degrees $\geq s$.

v_2 periodicity

Theorem

If M is a Q_2 local $E(2)$-module then $\Omega M \simeq \Sigma^7 M$.

Proof.

This is a general fact. If $A = B \otimes E[Q]$ and M is an A-module which is free over B, then tensoring M with

$$0 \longleftarrow \mathbf{F}_2 \longleftarrow E[Q] \longleftarrow \Sigma^{|Q|}\mathbf{F}_2 \longleftarrow 0$$

shows that $\Omega M \simeq \Sigma^{|Q|} M$. □

Remark

In this context, it is easy to construct a Q-localization $M \longrightarrow L_Q M$, a Q-colocalization $C_Q \longrightarrow M$, and a Tate module $T_Q M$ which sits in a triangle $C_Q \longrightarrow T_Q M \longrightarrow L_Q M$.

$\mathcal{A}(2)$

In $\mathcal{A}(2)$ the homologies which determine stable isomorphism are

$$[q_0, q_1, q_2, q_3] = [Q_0 < Q_1 < P_2^1 < Q_2].$$

The Margolis localizations and colocalizations they determine are

$$\begin{array}{ccccccccc}
M & = & M[0\text{–}3] & \leftarrow & M[0\text{–}2] & \leftarrow & M[0\text{–}1] & \leftarrow & M[0] \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & M[1\text{–}3] & \leftarrow & M[1\text{–}2] & \leftarrow & M[1] & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & M[2\text{–}3] & \leftarrow & M[2] & & & & \\
 & & \downarrow & & & & & & \\
 & & M[3] & & & & & &
\end{array}$$

v_2^8 periodicity

Theorem

If M is a Q_2-local $\mathcal{A}(2)$-module then $\Omega^8 M \simeq \Sigma^{56} M$.

Proof.

There is a complex

$$0 \longleftarrow \mathbf{F}_2 \longleftarrow P_0 \longleftarrow P_1 \longleftarrow P_2 \longleftarrow P_3$$

$$\longleftarrow P_4 \longleftarrow P_5 \longleftarrow P_6 \longleftarrow P_7 \longleftarrow \Sigma^{56}\mathbf{F}_2 \longleftarrow 0$$

in which each of the P_i is Q_2-acyclic.

□

Let us write $\mathcal{A}(2)$ as $\mathcal{A}$ for brevity.

- $P_0 = \mathcal{A}/(Sq^1, Sq^2)$ so that
$$\mathrm{Ext}_{\mathcal{A}}(P_0, \mathbf{F}_2) = \mathrm{Ext}_{\mathcal{A}(1)}(\mathbf{F}_2, \mathbf{F}_2) \Rightarrow \mathrm{ko}_*$$
- $P_1 = \Sigma^4\mathcal{A}/(Sq^1, Sq^2Sq^3)$ giving
$$\mathrm{Ext}_{\mathcal{A}}(P_1, \mathbf{F}_2) = \mathrm{Ext}_{\mathcal{A}(1)}(\Sigma^4\mathbf{F}_2, \mathbf{F}_2) \Rightarrow \Sigma^4\mathrm{ksp}_*$$
- $P_2 = \Sigma^8\mathcal{A}/(Sq^1)$ giving
$$\mathrm{Ext}_{\mathcal{A}}(P_1, \mathbf{F}_2) = \mathrm{Ext}_{\mathcal{A}(0)}(\Sigma^8\mathbf{F}_2, \mathbf{F}_2) \Rightarrow \Sigma^8\mathrm{H}\mathbb{Z}_*$$
- $P_3 = \Sigma^{15}\mathcal{A} \oplus \Sigma^{18}\mathcal{A}/(Sq^1, 0), (Sq^3, 0), (Sq^4, Sq^1), (Sq^4Sq^2, Sq^3)$ so that
$$0 \longleftarrow \Sigma^{18}\mathcal{A}/\!\!/E[Q_0, Q_1] \longleftarrow P_3 \longleftarrow \Sigma^{15}\mathcal{A}/\!\!/E[Q_0, Q_1] \longleftarrow 0$$
Thus, $\mathrm{Ext}_{\mathcal{A}}(P_3, \mathbf{F}_2)$ can be calculated as an extension of two copies of the Adams spectral sequence converging to ku_*.

- $P_4 = (\Sigma^{22}\mathcal{A} \oplus \Sigma^{24}\mathcal{A}) \,/\, \langle (Sq^1, 0), (Q_1, Sq^1), (0, Q_1)\rangle$, giving the first Adams cover of ku.
- $P_5 = \Sigma^{26}\mathcal{A}/(Sq^1) \oplus \Sigma^{24}\mathcal{A}/(Sq^2)$, giving an $\mathrm{H}\mathbb{Z}$ and the first Postnikov cover of ko.
- $P_6 = \Sigma^{33}\mathcal{A}/(Sq^1, Q_1) \oplus \Sigma^{36}\mathcal{A}/(Sq^{(0,2)})$, giving a ku and the bottom edge of a wedge: Ext of the second summand is $\mathbf{F}_2[\sqrt[4]{g}]$, polynomial on a class in $(t-s, s) = (5, 1)$ whose fourth power is g.
- $P_7 = (\Sigma^{39}\mathcal{A} \oplus \Sigma^{39}\mathcal{A}) \,/\, \langle (Sq^1, Sq^1), (Q_1, 0), (Sq^{(0,2)}, 0), (0, Sq^2)\rangle$,

$$0 \longleftarrow \Sigma^{39}\mathcal{A}/\!\!/\mathcal{A}(1) \longleftarrow P_7 \longleftarrow \Sigma^{39}\mathcal{A}/\!\!/E[Sq^{(0,1)}, Sq^{(0,2)}] \longleftarrow 0$$

Thus, $\mathrm{Ext}_{\mathcal{A}}(P_7, \mathbf{F}_2)$ can be calculated as an extension of the Adams spectral sequence converging to ko_* and the ‘wedge’, which has $\mathrm{Ext} = \mathbf{F}_2[v_1, \sqrt[4]{g}]$.

Advertisement - FPMods

- A sage package for doing calculations with finitely presented $\mathcal{A}$-modules.
- Written by Mike Catanzaro as his master's thesis.
- Allowed to become moribund, as new versions of sage broke some things.
- Recently resuscitated by Sverre Lunøe-Nielsen.
- Here is the sage code that verifies my claim about P_7:

```
sage: A2 = SteenrodAlgebra(p=2,profile=(3,2,1))
sage: load('fpmods.py')
sage: P7 = FP_Module([0,0],
             [[Sq(1),Sq(1)],[Sq(0,1),0],[Sq(0,2),0],[0,Sq(2)]]
     algebra=A2)
sage: ko = FP_Module([0],[[Sq(1)],[Sq(2)]],algebra=A2)
sage: p = FP_Hom(P7,ko,[[0],[1]])
sage: K,j = p.kernel()
sage: K.degs
```

Calculating $\mathrm{Ext}_{\mathcal{A}(2)}$

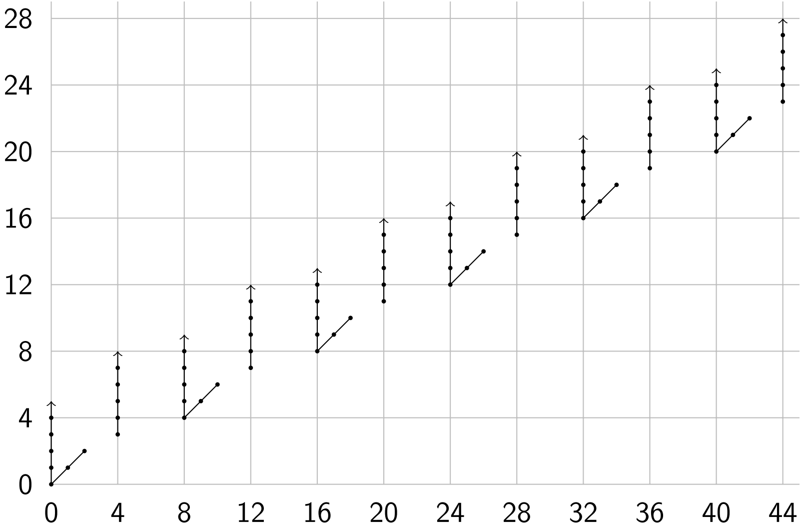


Figure: P_0

Calculating $\mathrm{Ext}_{\mathcal{A}(2)}$

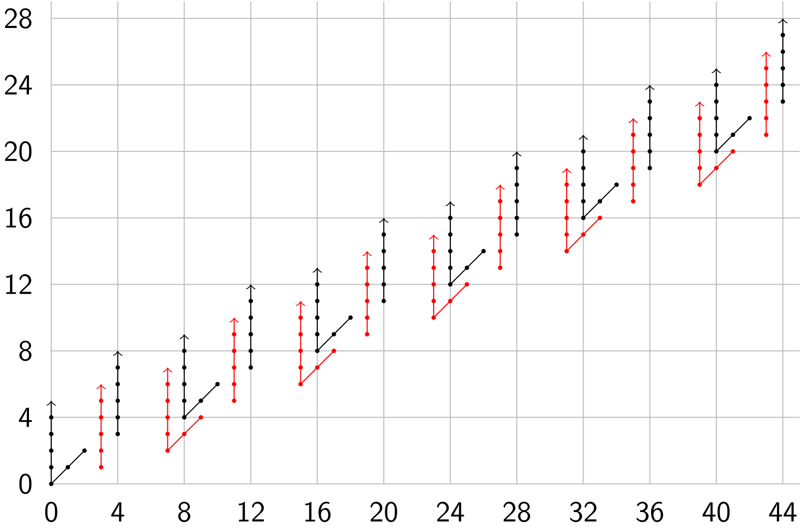


Figure: P_0, P_1

Calculating $\text{Ext}_{\mathcal{A}(2)}$

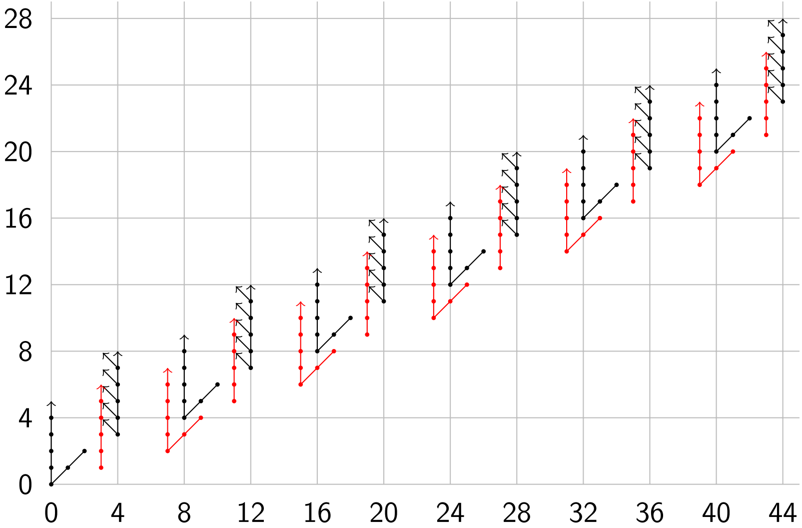


Figure: $P_0 \xrightarrow{d_1} P_1$

Calculating $\text{Ext}_{\mathcal{A}(2)}$

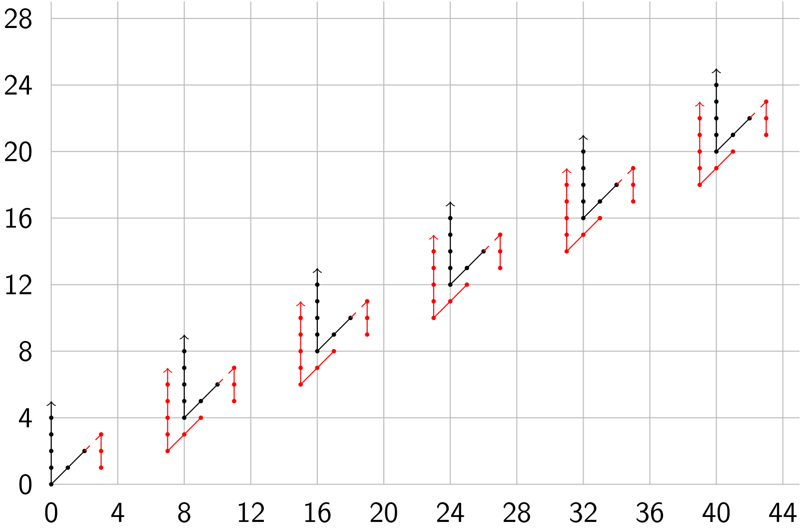


Figure: $H(P_0 \xrightarrow{d_1} P_1)$

Calculating $\mathrm{Ext}_{\mathcal{A}(2)}$

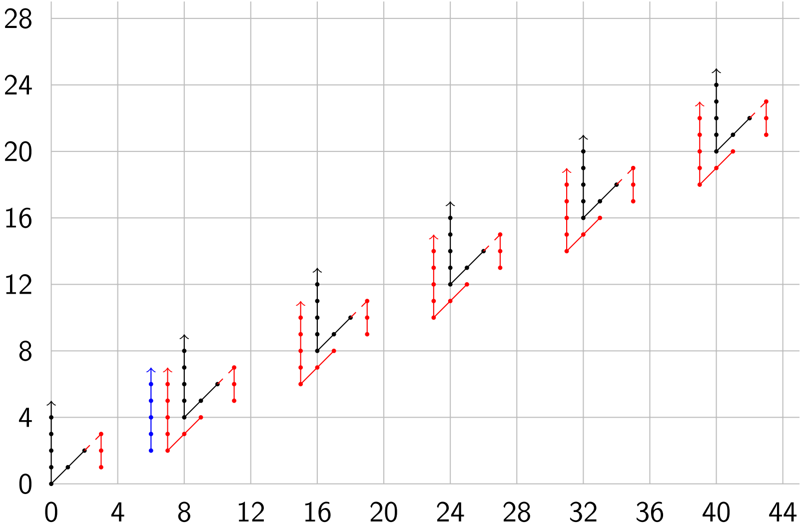


Figure: $H(P_0 \xrightarrow{d_1} P_1)$, P_2

Calculating $\text{Ext}_{\mathcal{A}(2)}$

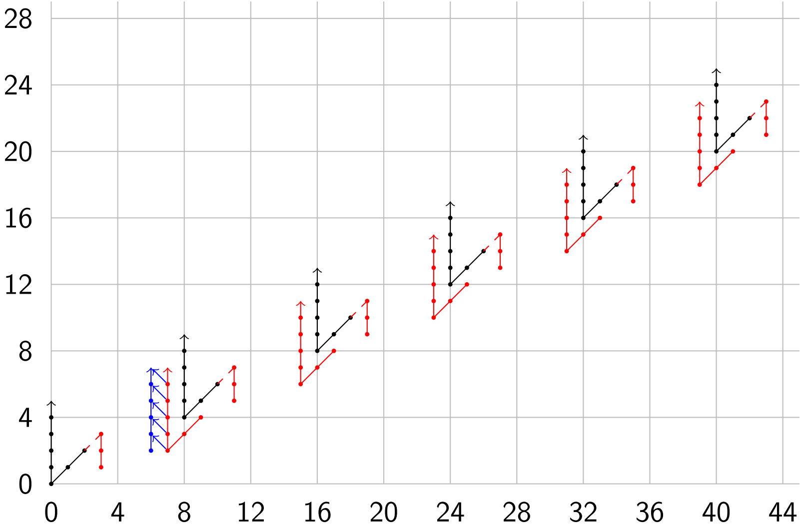


Figure: $H(P_0 \xrightarrow{d_1} P_1) \xrightarrow{d_1} P_2$

Calculating $\mathrm{Ext}_{\mathcal{A}(2)}$

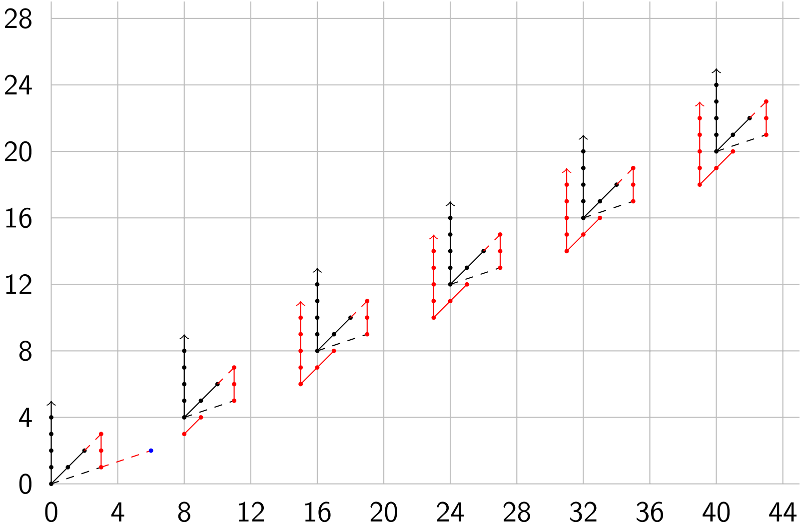


Figure: $H(P_0 \xrightarrow{d_1} P_1 \xrightarrow{d_1} P_2)$

Calculating $\mathrm{Ext}_{\mathcal{A}(2)}$

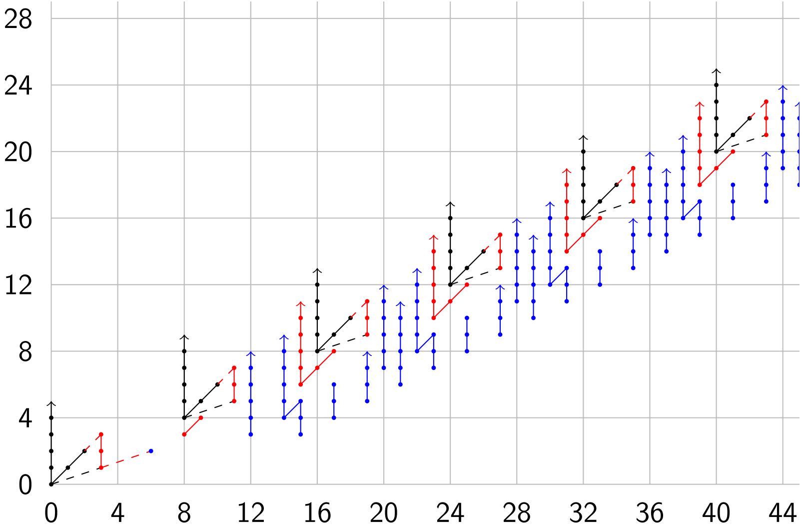


Figure: $H(P_0 \xrightarrow{d_1} P_1 \xrightarrow{d_1} P_2)$, P_3

Calculating $\text{Ext}_{\mathcal{A}(2)}$

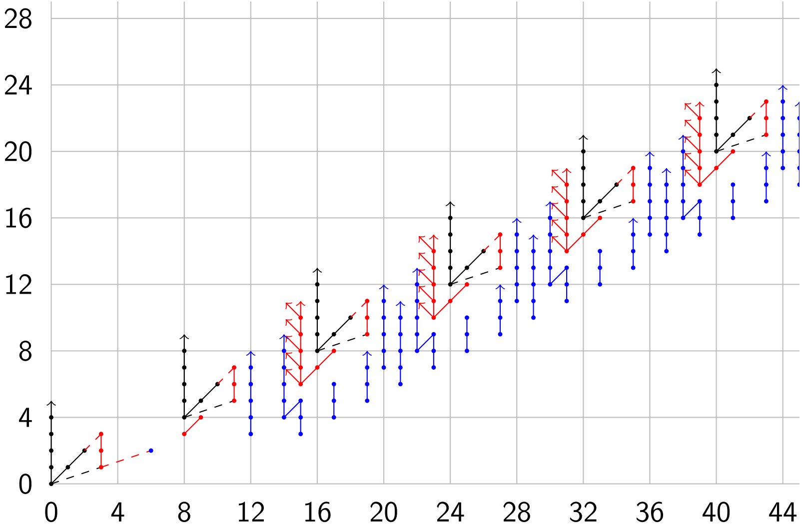


Figure: $H(P_0 \xrightarrow{d_1} P_1 \xrightarrow{d_1} P_2) \xrightarrow{d_2} P_3$

Calculating $\mathrm{Ext}_{\mathcal{A}(2)}$

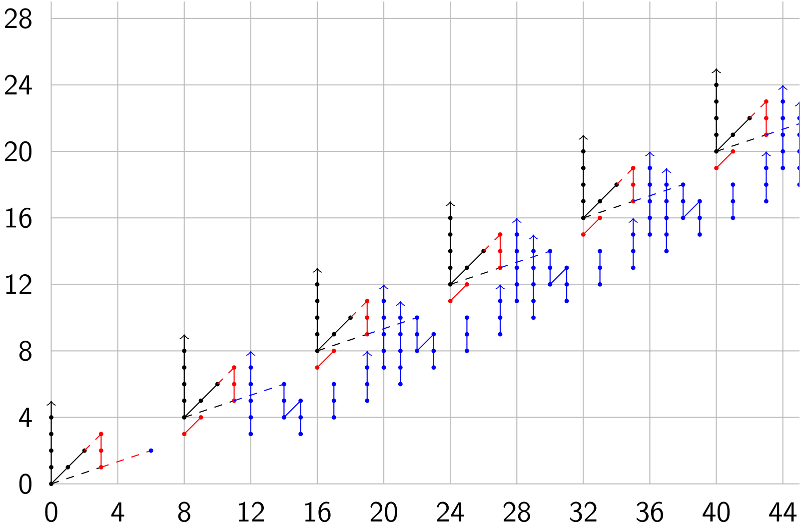


Figure: $E_3(P_0 \xrightarrow{d_1} P_1 \xrightarrow{d_1} P_2 \xrightarrow{d_2} P_3)$

Calculating $\mathrm{Ext}_{\mathcal{A}(2)}$

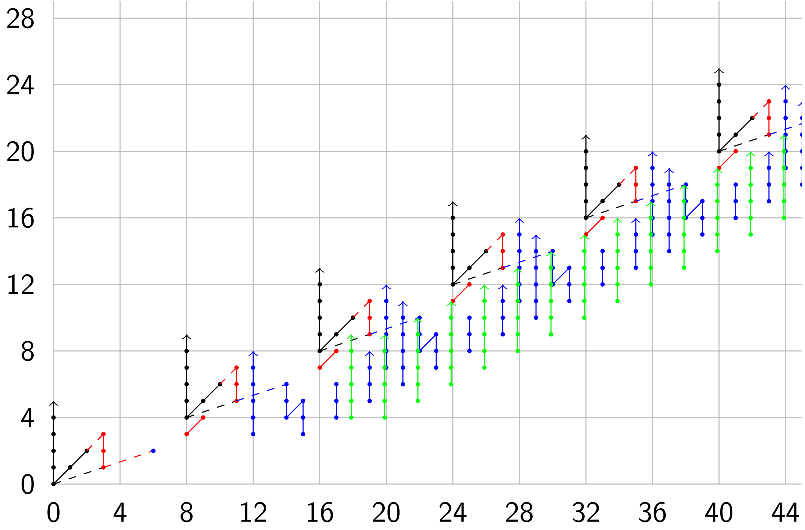


Figure: $E_3(P_0 \xrightarrow{d_1} P_1 \xrightarrow{d_1} P_2 \xrightarrow{d_2} P_3),\ P_4$

$\mathrm{Ext}_{\mathcal{A}(2)}$

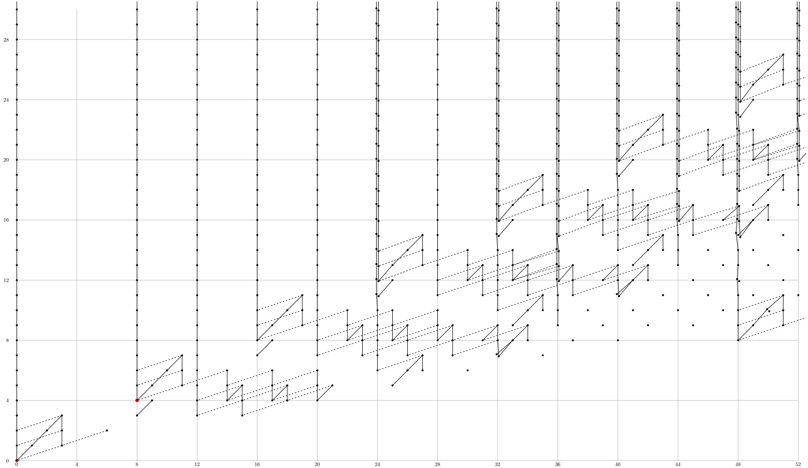


Figure 1: $\mathrm{Ext}_{A(2)}$, $0 \le n \le 52$, $0 \le s \le 30$

Similar calculations

- There is a v_2^4 sequence for $H^*(C(\nu))$ giving a simple calculation of $\mathrm{Ext}_{\mathcal{A}(2)}(H^* C\nu, \mathbf{F}_2)$.
- v_2^8 sequences with small, well understood terms, exist for the cofibers of 2 and η, as well as other complexes useful in computing $THH_*(tmf)$.

$\mathrm{Ext}_{\mathcal{A}(2)}$ again

Theorem (Iwai and Shimada)

The cohomology of $A(2)$ is

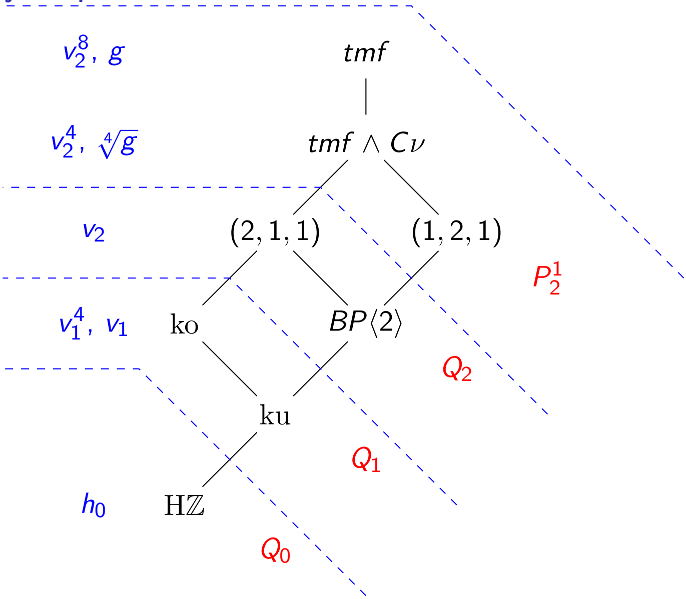
$$\mathrm{Ext}_{A(2)}(\mathbf{F}_2, \mathbf{F}_2) = \mathbf{F}_2[h_0, h_1, h_2, c_0, d_0, e_0, g, \alpha, \beta, \gamma, \delta, w_1, w_2]/I.$$

The ideal I has 56 generators:

- $h_0 h_1,\ h_1 h_2,\ h_0^2 h_2 - h_1^3,\ h_0 h_2^2,\ h_2^3$
- $\ldots$
- $c_0\gamma - h_1\delta,\ \beta\gamma - g^2,\ d_0^2 - g w_1,\ \gamma\delta - h_1 c_0 w_2,$
- $\gamma^2 - h_1^2 w_2 - g\beta^2,\ \alpha^4 - h_0^4 w_2 - w_1 g^2$

- Free over $\mathbf{F}_2[w_1, w_2]$; here w_1 and w_2 restrict to v_1^4 and v_2^8, resp.
- A sum of cyclic $R = \mathbf{F}_2[g, w_1, w_2]$-modules isomorphic to R, $R/(g)$ and $R/(g^2)$.
- Four infinite families, $h_0^i \alpha^j$, $i \geq 0$, $0 \leq j \leq 3$.
- Thirty-two other summands.
- E_3, E_4 and $E_5 = E_\infty$ are then modules over $R_1 = \mathbf{F}_2[g, w_1, w_2^2]$ and $R_2 = \mathbf{F}_2[g, w_1, w_2^4]$ resp. Mostly cyclic.

Summary of periodicities



E_∞

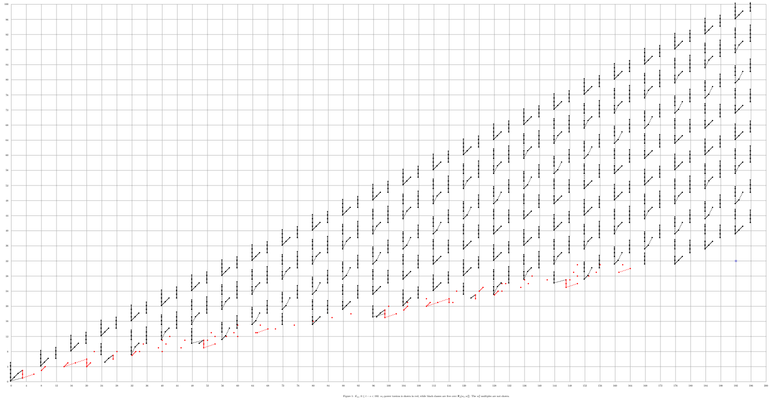


Figure: E_∞, $0 \leq t - s < 192$. w_1-power torsion is shown in red, while black classes are free over $\mathbf{F}_2[w_1, w_2^4]$. The w_2^4 multiples are not shown.

Tusen takk